Mathematical Reviews

Vol. 17, No. 2 FEBRUARY 1956 pp. 117-222

HISTORY, BIOGRAPHY

★**d'Ocagne, Maurice. Histoire abrégée des sciences mathématiques.** Ouvrage recueilli et achevé par René Dugas. Librairie Vuibert, Paris, 1955. 405 pp. 1350 francs.

Enumération très conventionelle de mathématiciens et de leurs ouvrages, extrait des manuels historiques du 19ième siècle, à quelques additions d'origine inconnue près. Dans „la période contemporaire" le récit sur l'algèbre va comme il suit: Montel reprend les propriétés de Lucas-Gauss et de Landau et précise la grandeur des racines d'une équation d'après la densité des termes de celle-ci; il étudie les rapports entre les racines des équations et des dérivées, et les fonctions rationelles enlacées. De ce point du vue sont issus les travaux de Favard, Dieudonné, Biernacki, Tchacaloff. Pour la théorie des nombres on ne lit que des noms français. Les topologistes connus de l'auteur sont Lebesque, Antoine, Marchaud, Zoretti, Hilbert, Brouwer, Alexander, Errera. Le seul constructeur de machines à calculer est Couffignal. La liste des „mathématiciennes" finit par Kowalewska.

H. Freudenthal (Utrecht).

Gillings, R. J. The oriental influence on Greek mathematics. Math. Gaz. 39 (1955), 187–190.

Schmidt, Olaf. On the theorems of Ptolemy and Menelaus. Nordisk Mat. Tidskr. 3 (1955), 81–95, 127. (Danish. English summary)

Irani, Rida A. K. Arabic numeral forms. Centaurus 4 (1955), 1–12.

★**Delone, B. N. Matematika i ee razvitie v Rossii. [Mathematics and its development in Russia.]** Izdat. "Pravda", Moscow, 1948. 16 pp.

Koyré, Alexandre. A documentary history of the problem of fall from Kepler to Newton. De motu gravium naturaliter cadentium in hypothesi terrae motae. Trans. Amer. Philos. Soc. (N.S.) 45 (1955), 329–395.

★**Freudenthal, H. The concept of space in the exact sciences from Kant up to the present.** Handelingen van het XXXIV^e Nederlands Natuur- en Geneeskundig Congres, Wageningen, 1955, pp. 82–95. (Dutch)

Popken, J. Un aperçu historique sur les nombres transcendants. Bull. Soc. Math. Belg. 1954, 71–82 (1955).

Brun, Viggo. Découverte d'un manuscrit d'Abel. Rev. Hist. Sci. Appl. 8 (1955), 103–106.

Shankar Shukla, Kripa. Ācārya Jayadeva, the mathematician. Ganita 5 (1954), 1–20.

★**Tóth, Imre. Johann Bolyai. Leben und Werk des grossen Mathematikers.** Technischer Verlag, Bukarest, 1955. 73 pp. 0.75 Lei.

Frank, Philipp G. Einstein. Synthese 9 (1954), 435–437.

Isnardi, Teofilo. Albert Einstein. An. Soc. Ci. Argentina 159 (1955), 3–8 (1 plate). (Spanish)

Petiau, Gérard. Albert Einstein, 1879-1955. Rev. Gén. Sci. Pures Appl. 62 (1955), 227–236.

Fokker, A. D. Albert Einstein, inventor of chronogeometry. Synthese 9 (1954), 442–444.

Mannoury, G. The cultural phenomenon Albert Einstein. Synthese 9 (1954), 438–441.

★**Wolfer, Ernst Paul. Eratosthenes von Kyrene als Mathematiker und Philosoph.** Dissertation, Universität Zürich, 1954. P. Noordhoff N.V., Groningen-Djakarta, 1954. iv+68 pp. $1.00.

Latyševa, K. Ya. On the works of V. P. Ermakova (1845-1922) on the theory of ordinary differential equations. Ukrain. Mat. Ž. 7 (1955), 231–238. (Russian)

Hofmann, J. E. Carl Friedrich Gauss. Lebensskizze zum Gedächtnis an die 100. Wiederkehr seines Todestages. Math. Naturwiss. Unterricht 8 (1955), 49–60.

Popken, J. Some facets of Gauss's work. Euclides, Groningen 30 (1954/55), 282–292. (Dutch)

Sofonea, Traian. Gauss und die Versicherung. Hundert Jahre nach dem Tode des "princeps mathematicorum" (1777-1855). Verzekerings-Arch. Actuarieel Bijvoegsel 32 (1955), 57*–69* (1 plate).

Gauss as geometer. Euclides, Groningen 30 (1954/55), 276–281. (Dutch)

Vollgraff, J. A. Kurze Bemerkungen, und Zitate, über Vergangenheit und Zukunft der Mathematik, so wie Gino Loria (1862-1954) versucht hat sich dieselben zu denken. Synthese 9 (1954), 485–491.

Cath, P. G. Jules Henri Poincaré (Nancy 1854-Paris 1912). Euclides, Groningen 30 (1954/55), 265–275. (Dutch)

Natucci, Alpinolo. Nel primo centenario della nascita di Enrico Poincaré. Giorn. Mat. Battaglini (5) 3(83) (1955), 115–130.

Pajares, E. Simon Stevin. Gac. Mat., Madrid (1) **7** (1955), 3–6 (1 plate). (Spanish)

Carmody, Francis J. Notes on the astronomical works of Thâbit b. Qurra. Isis **46** (1955), 235–242.

Tenca, Luigi. Giovanni Wallis e gli italiani. Boll. Un. Mat. Ital. (3) **10** (1955), 412–418.

Obituary: Tibor Szele. Publ. Math. Debrecen **3** (1954), 193–194 (1955).

FOUNDATIONS, MATHEMATICAL LOGIC

Stanley, Robert L. Simplified foundations for mathematical logic. J. Symb. Logic **20** (1955), 123–139.

Verf. legt eine explizite Formulierung seiner Version SF des Quineschen Systems NF vor, die schon 1952 angekündigt war. Es wird gezeigt, dass SF alle Theoreme von NF liefert und dass andrerseits SF widerspruchsfrei relativ zu NF ist.

SF benutzt nur Terme $\zeta, \eta, \vartheta, \cdots$, die als Klassen zu interpretieren sind. Mit Variablen $\alpha, \beta, \cdots$ für Klassen, wird $(\zeta/\alpha/\eta)$ als einziger Grundbegriff eingeführt, der als $\hat{\alpha}(\zeta \cap \eta = \Lambda)$ zu interpretieren ist. Tritt α weder in ζ noch η auf, so ist $(\zeta/\alpha/\eta)$ als universelle bzw. leere Klasse, V bzw. Λ, zu interpretieren, wenn $\zeta \cap \eta = \Lambda$ wahr bzw. falsch ist. Statt $(\zeta/\alpha/\eta)$ wird dann (ζ/η) geschrieben und diese Terme $\varphi, \psi, \cdots$ sind als Formeln zu interpretieren, nämlich als $\varphi = V \cdot (\varphi/\psi)$ ist als $\sim(\varphi \cdot \psi)$ zu interpretieren und es lassen sich dementsprechend die aussagenlogischen Partikeln $\sim, \cdot, \vee, \supset$ mit Hilfe von / definieren. $(\alpha)\varphi$ wird definiert durch $\sim(\varphi/\alpha/\varphi)$, denn dies ist zu interpretieren als $\alpha(\varphi = \Lambda) = \Lambda$, also $\hat{\alpha}(\varphi = V) = V$. $\hat{\alpha}\varphi$ wird definiert durch $(\sim\varphi/\alpha/\sim\varphi)$, denn dies ist als $\hat{\alpha}(\sim\varphi = \Lambda)$, d. h. als $\hat{\alpha}(\varphi = V)$ zu interpretieren. $\zeta \subset \eta$ wird definiert durch $(\alpha)(\alpha/\eta \supset \alpha/\zeta)$ und anschliessend lässt sich ϵ wie bei Quine definieren. Neben ϵ wird noch $\zeta\, e\, (\eta(\alpha)/\alpha\vartheta(\alpha))$ als Abkürzung für $(\eta(\zeta)/\vartheta(\zeta))$ eingeführt. Die Stratifikation kommt dadurch herein, dass aus (ζ/η) nur für stratifizierte ϑ auf $\vartheta\, \bar{\epsilon}\, \zeta \vee \vartheta\, \bar{\epsilon}\, \eta$ geschlossen werden darf. Die (wenigen) Axiome und Regeln können hier nicht referiert werden; jedenfalls kann aus $\zeta \epsilon \hat{\alpha}\varphi(\alpha)$ auf $\varphi(\zeta)$ geschlossen werden, wenn ζ stratifiziert ist — im Unterschied zu NF, wo $\varphi(\alpha)$ stratifiziert sein muss. Verf. hebt hervor, dass die Deduktionen in SF im wesentlichen die Züge der „natürlichen" Deduktion haben. *P. Lorenzen* (Bonn).

Reichbach, Juliusz. Über den auf Alternative und Negation aufgebauten Aussagenkalkül. Studia Logica **1** (1953), 13–18 (1954). (Polish and Russian summaries)

The author exhibits a system of propositional calculus based on alternation (A) and negation (N) having the usual truth tables. The axioms and rules of the system are the following. Rules: a*) Rule of substitution. b*) Rule of detachment, given by the scheme:

$$\frac{A\alpha\beta \quad N\alpha}{\beta}.$$

Axioms: I*) $NNApANpq$; II*) $NNANApqANANqpp$; III*) $NNANApqANANqrApr$; IV*) $ANpp$; V*) $ANApqApNNq$.

The independence and completeness of the system is demonstrated. It is shown that every theorem of a system in which all truth-table valid propositional forms are derivable is a theorem of the above system. This property of the second system is stated without proof, and its axioms, listed below, are said to be independent. Rules: a) Rule of substitution. b) Rule of detachment, given by the scheme:

$$\frac{AN\alpha\beta \quad \alpha}{\beta}.$$

Axioms: I) $ApANpq$; II) $ANApqANANqpp$; III) $ANApqANANqrApr$.

Erratum: Page 15, first line of proof of Theorem II: read T_A for T_A*. *E. J. Cogan* (Hanover, N.H.).

Myhill, John. Creative sets. Z. Math. Logik Grundlagen Math. **1** (1955), 97–108.

Let a non-negative integer be called a number and let a collection of numbers be called a set. The recursively enumerable (r.e.) set α is said to be represented in the (formal) system Σ if Σ contains a formula $\Phi(n)$ with one free variable n such that n belongs to α if and only if $\Phi(n)$ is provable in Σ. The system Σ_1 is said to be translatable into the system Σ_2 if there is an effective method whereby, given any formula Φ_1 of Σ_1, we can find a formula Φ_2 of Σ_2, such that Φ_1 is provable in Σ_1 just in case Φ_2 is provable in Σ_2. The systems Σ_1 and Σ_2 are called notational variants of each other if there exists an effective one-to-one correspondence between the collection of all formulas of Σ_1 and the collection of all formulas of Σ_2 under which the theorems of Σ_1 correspond to the theorems of Σ_2 and the non-theorems of Σ_1 to the non-theorems of Σ_2. This paper is concerned with the class of all undecidable systems whose undecidability can be established by the use of one of the following two theorems: (A) A system in which every r.e. set is represented is undecidable; (B) if Σ_0 is undecidable and translatable into Σ, then Σ is likewise undecidable. Most systems which are known to be undecidable belong to this class. The principal result of this paper is that any two undecidable systems whose undecidability can be established using (A) or (B) are notational variants of each other.

A one-to-one recursive function from the set of all numbers onto itself is called a recursive permutation. Two sets are said to be isomorphic if one can be mapped onto the other by some recursive permutation. The set α is called one-to-one reducible to the set β if $x \epsilon \alpha \leftrightarrow f(x) \epsilon \beta$, for some one-to-one recursive function $f(x)$. Let all r.e. sets be effectively generated in the sequence $\omega_0, \omega_1, \cdots$. Then α is called creative if α is r.e. and there exists a partial recursive function $f(n)$ such that $\omega_n \subset \alpha'$ implies: (1) $f(n)$ is defined, (2) $f(n) \notin \alpha + \omega_n$. This concept was first introduced by E. L. Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29], though in a slightly different manner. The following four theorems lead to the principal result mentioned above. Let Σ be any system whose undecidability is provable by the use of theorem (A) or (B); then the set of all Gödel numbers of the theorems in Σ is creative. Any two creative sets are one-to-one reducible to each other. If two sets are one-to-one reducible to each other they are isomorphic. Any two creative sets are isomorphic. *J. C. E. Dekker* (Chicago, Ill.).

Montague, Richard. On the paradox of grounded classes. J. Symb. Logic **20** (1955), 140.

The author states a paradox equivalent to Yuting's paradox of the class of all grounded classes [same J. **18** (1953), 114; MR **15**, 91], in a form which avoids the notions of sequence and natural number. He calls a class x regular if $(k)(x \epsilon k \supset (\exists y)(y \epsilon k \,.\, \sim(\exists z)(z \epsilon k \,.\, z \epsilon y)))$, and shows

that the class of all regular classes is neither regular nor non-regular. Using the axiom of choice it can be shown that the regular classes are just Yuting's grounded classes.
O. Frink (University Park, Pa.).

Kreisel, G. Remark on complete interpretations by models. Arch. Math. Logik Grundlagenforsch. **2** (1954), 4–9.

To every formula $\mathfrak{A}$ of the second-order predicate calculus Π a formula $\mathfrak{A}_1$ of the first-order predicate calculus Π_1 is associated; by definition, $\mathfrak{A}$ is provable in Π if $\mathfrak{A}_1$ is provable in Π_1. $\mathfrak{A}_1$ is constructed in such a way that, intuitively, Π is equivalent to the usual second-order calculus. In order to extend the results of Hilbert and Bernays, Grundlagen der Mathematik [Bd. II, Springer, Berlin, 1939, pp. 234–253] to formulas in Π, the author constructs a sequence $D^{(n)}$ of domains of integers and a sequence $D_p^{(n)}$ of sets of predicates; let A_n be the formula got by restricting the individual variables of $\mathfrak{A}$ to $D^{(n)}$ and the predicate variables to $D_p^{(n)}$; A_n can be written as a formula of Z_μ [Hilbert and Bernays, loc. cit., p. 293]. If A can be proved in Π, each A_n can be proved in Z_μ; if A cannot be proved in Π, there is an A_n which can be disproved in $\mathfrak{Z}$ [Hilbert and Bernays, loc. cit., p. 352].
A. Heyting (Amsterdam).

Hintikka, K. Jaakko J. Reductions in the theory of types. Acta Philos. Fenn. **8** (1955), 57–115.

By introducing suitable auxiliary predicate symbols, many-sorted predicate calculi can be reduced to the predicate calculus of first order (pc1) [A. Schmidt, Math. Ann. **123** (1951), 187–200; MR **13**, 614]. In a similar way the author (a) associates with each formula K of the simple theory of finite types (stt) a formula $\xi(K)$ of pc1 where $\xi(\sim K)$ is $\sim\xi(K)$ and (ii) constructs a formula D, with one bound monadic predicate variable, which expresses that the auxiliary predicates represent the notions of stt as intended. He shows that $D \supset \xi(K)$ is valid if and only if K is valid in stt. This establishes the very plausible result that, from the point of view of validity, stt can be reduced to pc2. In his treatment the axioms of class formation are stated informally in terms of arbitrary classes of symbols, which accounts for the bound predicate variable in D. For formalized axioms of class formation the author sketches a method of replacing D by a formula of pc1. This answers a question raised by the reviewer, in the paper reviewed above, in connection with his derivation of Henkin's completeness theorem for stt [J. Symb. Logic **15** (1950), 81–91; MR **12**, 70] from the completeness of pc1.
G. Kreisel (Princeton, N.J.).

Yonemitsu, Naoto. A decision method and a topological interpretation for systems of logical implication. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **1954**, no. 3, 6–20.

The author gives a detailed treatment of several variants of Lewis' modal logic along the lines of McKinsey [J. Symb. Logic **6** (1941), 117–134; MR **3**, 290].
G. Kreisel (Princeton, N.J.).

Yonemitsu, Naoto. A note on systems of logical implication. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **1954**, no. 3, 21–24.

By the use of suitable matrices the author establishes the independence of the primitive symbols in the systems of modal logic considered in the paper reviewed above.
G. Kreisel (Princeton, N.J.).

Fröhlich, A., and Shepherdson, J. C. On the factorisation of polynomials in a finite number of steps. Math. Z. **62** (1955), 331–334.

This is a resumé of a longer paper to appear later. Modifying a construction of van der Waerden [Math. Ann. **102** (1930), 738–739], the authors obtain a recursive enumeration of the elements of (i) a field K_0 such that there is no recursive algorithm for deciding whether a polynomial with coefficients in K_0 is reducible (splitting algorithm); (ii) a field K which has a splitting algorithm yet $K(\sqrt{x_0})$ does not. A second representation of $K(\sqrt{x_0})$ is given which does have a splitting algorithm: the isomorphism between the two representations is, of course, not recursive. The authors have examined the literature on splitting algorithms in order to determine which of the relevant theorems are concerned with particular representations of the fields considered. Finally they criticize Krull's distinction [Math. Z. **59** (1953), 57–60; MR **15**, 96] between splitting algorithms in a narrow and an extended sense, because this distinction is irrelevant from the point of view of high-speed computing machines or intuitionist objections to the law of excluded middle. In the reviewer's opinion Krull's examples make it quite clear that he was never concerned with these points of view. He evidently wanted to avoid the use of auxiliary notions in the algorithm which are not 'directly' concerned with divisibility properties of the coefficients such as, e.g., the number of prime factors. Krull's somewhat vague notion may perhaps be made precise along the lines of the reviewer's definition of 'direct' proofs in number theory [Actes 2e Colloq. Internat. Logique Math., Paris, 1952, Gauthier-Villars, Paris, 1954, pp. 37–49; MR **16**, 782].
G. Kreisel.

Rubin, Herman, and Suppes, Patrick. A note on two-place predicates and fitting sequences of measure functions. J. Symb. Logic **20** (1955), 121–122.

Verf. weisen mit Hilfe einer einfachen gruppentheoretischen Betrachtung nach, dass das Carnapsche Mass m^* einer Aussage von der Anzahl der Individuen, die in der Sprache auftreten, abhängig werden kann, falls man auch 2-stellige Prädikate (statt nur 1-stelliger) zulässt.
P. Lorenzen (Bonn).

Davidson, Donald, McKinsey, J. C. C., and Suppes, Patrick. Outlines of a formal theory of value. I. Philos. Sci. **22** (1955), 140–160.

A discussion of the formal properties of value, or rather of (subjectivistic) utility. The discussion is mainly expository but contains several ideas that seem novel. It is convincingly argued that an axiomatization similar to that of von Neumann and Morgenstern [Theory of games and economic behaviour, 2nd ed., Princeton, 1947; MR **9**, 50] can be justified for rational judgements of utilities. A basic assumption [or partial definition of rationality] is that a body of preferences among a set K of alternatives is irrational if there exists a subset [presumably finite] L, of K, in which no rational choice exists. Here a rational choice (in L) is a choice that selects an alternative to which none is preferred.

There is a list of 25 references and a statement that further work is being carried out by the Stanford Value-Theory Project.

It is contended, and the reviewer would agree, that more philosophers could learn something of value about value from this kind of work, which is partly philosophical and partly mathematical and is traditionally associated with welfare economics.
I. J. Good (Cheltenham).

Huzurbazar, V. S. On the certainty of an inductive inference. Proc. Cambridge Philos. Soc. **51** (1955), 761–762.

It is proved that "repeated verification of the consequences of a hypothesis with non-zero prior probability will make it almost certain that any number of further consequences of it will be verified." This extends a weaker conclusion of H. Jeffreys that refers only to any single further consequence.

Mathematically, the theorem says that if F_i is an infinite sequence of events, $F_i \supset E$, $P(E)>0$, then

$$\lim_{\substack{n\to\infty\\ m\to\infty}} P\Big(\bigcap_{n+1}^{n+m} F_i \Big| \bigcap_{1}^{n} F_i\Big)=1.$$

The proof is short and straightforward. *L. J. Savage.*

Sundrum, R. M. The logic of inference in statistical theory. J. Burma Res. Soc. **37** (1954), 82–89.

Freudenthal, Hans. The concepts of axiom and axiomatics in mathematics and physics. Simon Stevin **30** (1955), 156–175. (Dutch)

The author discusses both the conceptions about axioms and axiomatics in Greek mathematics and philosophy and their recent development. On the present situation his comments are stimulating but controversial. Freudenthal rejects the distinction between mathematics and metamathematics which, in his opinion, has been successfully transgressed by Brouwer in his proof of the Fundamental Theorem on Finitary Spreads; however, he seems to underestimate achievements in the field of foundations. *E. W. Beth* (Amsterdam).

Bernays, Paul. Die Mathematik als ein zugleich vertrautes und unbekanntes. Synthese **9** (1954), 465–471.

See also: Grünbaum, p. 202.

ALGEBRA

Linear Algebra, Polynomials

Levit, R. J. Some linear minimax problems over an ordered field. Amer. J. Math. **77** (1955), 541–562.

Variables x, parameters t, and functions f (linear in x and t) are finite in number. The system $f=0$ is a mixed representation of a linear variety L in x-space which includes as special cases the parametric and the parameter-free representation. The points on L are determined for which the absolutely largest coordinate is a minimum μ, and it is shown that $\mu=\alpha_0/\sum\alpha_k$, where the α_k are certain minors of the coefficient matrix. The result is also specialized to the parametric case. Applications to Chebyshev approximation are discussed. (Remark: Theorem 2 on skew-symmetric matrices A of order two can be completed by writing $A=B'CB$, $C=\begin{pmatrix}0 & c\\ -c & 0\end{pmatrix}$; then two rows of A contain no minor with terms of opposite sign if and only if the corresponding values b_{2i}/b_{1i} are identical or, among all such values, cyclically adjacent on the projective line.)
T. S. Motzkin (Los Angeles, Calif.).

Garaj, Jozef. Contribution to the representation of a vector algebra in a Minkowski four-dimensional space. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 22–38. (Slovak. Russian summary)

Dans le travail l'auteur définit deux produits des deux vecteurs dans l'espace de Minkowski: Le produit „antisymétrique" et le produit „complémentaire" et montre une certaine dualité. Le travail est méthodologique.
F. Vyčichlo (Prague).

Astuni, Enrico. Sulla risoluzione dell'equazione quartica. Ricerca Sci. **25** (1955), 2295–2312.

The general quartic is reduced to

$$x^4+a_2x^2+a_3x+a_4=0$$

and then by setting $u=4a_4a_2^{-2}$, $t=a_3^2a_2^{-3}$ the cubic

$$m^3+2m^2+(1-u)m-t=0$$

is solved by means of a table [4 pages] for its real roots. The desired roots of the quartic are then obtained by solving quadratic equations. The table is quite inadequate for practical use. A number of simple examples are set forth in an appendix. *D. H. Lehmer* (Berkeley, Calif.).

Cecioni, Francesco. Alcune osservazioni sulla teoria della divisibilità. Boll. Un. Mat. Ital. (3) **10** (1955), 382–400.

Danguy, Louis. Quelques cas de résolution explicite d'une équation de degré N. Mathesis **64** (1955), 115–117.

Carlitz, L., and Olson, F. R. A problem in factorization of polynomials. Math. Scand. **3** (1955), 28–30.

Let $f(x)$ be a polynomial of degree n with coefficients in the rational field R and irreducible over R. If k is an arbitrary rational number different from zero, then $f(x+kf(x))=f(x)\phi(x)$, where $\phi(x)$ is a polynomial with rational coefficients. The authors are primarily concerned with the case when $f(x)$ is cubic and derive conditions that $\phi(x)$ factor into two cubics. They show that for a given cubic the number of choices for k may be zero, finite or infinite. *A. L. Whiteman* (Los Angeles, Calif.).

See also: Roşculeţ, p. 145; Dantzig, p. 171; Chao, p. 194.

Lattices

Szász, G. Generalization of a theorem of Birkhoff concerning maximal chains of a certain type of lattices. Acta Sci. Math. Szeged **16** (1955), 89–91; correction, 270.

The author gives the following example of a distributive lattice in which not all maximal chains between 0 and I have the same cardinal number. Consider the direct product of the closed intervals, $0\leq x_1\leq 1$, x_1 real, and $0\leq x_2\leq 1$, x_2 rational; take as the partial ordering: $(x_1, x_2)\geq(y_1, y_2)$ if and only if $x_1\geq y_1$ and $x_2\geq y_2$; finally, identify all the elements $(x_1, 1)$. This identification does not come from a congruence relation, so that the stated proof is incorrect. (The author writes that a corrected proof will appear soon in the same journal.)
S. Gorn (Philadelphia, Pa.).

Močul'skiĭ, E. N. Direct decompositions of Dedekind lattices. Mat. Sb. N.S. **37 (79)** (1955), 89–102. (Russian)

The author somewhat generalizes and extends previous work on direct decompositions of a modular lattice by

splitting endomorphisms, particularly the work of A. H. Livšic [Mat. Sb. N. S. **28(70)** (1951), 481–502; MR **13**, 100] and L. A. Hostinsky [Duke Math. J. **18** (1951), 331–342; Amer. J. Math. **73** (1951), 741–755; Proc. Amer. Math. Soc. **5** (1954), 315–319; MR **12**, 795; **13**, 525; **15**, 673]. Primarily, the previous assumption that the lattice is complete is replaced by the requirement that certain ascending chains have unions.

P. M. Whitman (Silver Spring, Md.).

Baer, R. M. A characterization theorem for lattices with Hausdorff interval topology. J. Math. Soc. Japan **7** (1955), 177–181.

E. S. Northam showed that a necessary condition that a lattice be a Hausdorff space in its interval topology is that each closed interval of the lattice have a finite separating set [Proc. Amer. Math. Soc. **4** (1953), 824–827; MR **15**, 244]. In this note the author shows by examples that this necessary condition is not sufficient. He gives a condition stronger than Northam's, which is shown to be both necessary and sufficient for the interval topology of a lattice to be Hausdorff. This condition requires that corresponding to every interval of the lattice L there exist two finite sets A and B of the interval such that certain half intervals determined by the elements of A and B cover L.

O. Frink (University Park, Pa.).

Wang, Shih-chiang. Notes on the permutability of congruence relations. Acta Math. Sinica **3** (1953), 133–141. (Chinese. English summary)

The following results are obtained. (1) The congruence relations on any relatively complemented lattice are permutable. (2) The congruence relations on a complemented modular lattice L form a Boolean algebra if and only if all neutral ideals of L are principal. (3) The congruence relations on any finite quasigroup are permutable. (4) For any infinite cardinal α, there exist loops of α elements with non-permutable congruence relations. (1) was conjectured by G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 86, Ex. 3; MR **10**, 673]. (3) and (4) together give a complete solution of Birkhoff's Problem 31 [ibid., p. 86]. The author was unaware that (3) has been given by G. Trevisan [Rend. Sem. Mat. Univ. Padova **19** (1950), 367–370; MR **12**, 313], who also proved that the normal congruence relations on any quasigroup are permutable.

Ky Fan (Notre Dame, Ind.).

Wang, Shih-chiang. A note on ordered rings of real vectors. Acta Math. Sinica **5** (1955), 65–80. (Chinese. English summary)

Let G be the ordered additive group of all real vectors (a, b), in which $(a, b) \geqq (0, 0)$ means $a>0$, or $a=0$ and $b \geqq 0$. Solving G. Birkhoff's Problem 103 [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 229; MR **10**, 673], the author proves that G can be made into an ordered ring by each of the following types of multiplication. I. $(a, b)(c, d)=(0, \beta ac)$, $\beta \geqq 0$ constant. II. $(a, b)(c, d)=(\alpha ac, g(ac))$, where $\alpha>0$ and g is any real function satisfying $g(x+y)=g(x)+g(y)$. III. $(a, b)(c, d)=(\alpha ac, f(a, c)+\alpha bc)$, where $\alpha>0$ and f is any real function of two real variables satisfying (1) $f(x+y, z)=f(x, z)+f(y, z)$, (2) $f(x, y+z)=f(x, y)+f(x, z)$ and (3) $f(xyz, u)-f(x, yzu)=zuf(x, y)$. IV. $(a, b)(c, d)=(\alpha ac, f(a, c)+\alpha ad)$, where $\alpha>0$ and f satisfies (1), (2) and (4) $f(xyz, u)-f(x, yzu)=xzf(y, u)$. V. $(a, b)(c, d)=(\alpha ac, f(a, c)+\alpha ad+\alpha bc)$, where $\alpha>0$ and f satisfies (1), (2) and (5) $f(xyz, u)-f(x, yzu)=xzf(y, u)-zuf(x, y)$. As example, one can take $f(x, y)=0$ in III and IV, and $f(x, y)=\beta xy$ (β any real constant) in V. Furthermore, I–V are all possible types of multiplication, if G is to become an ordered ring. Although the problem of determining the function f is left open, the author points out that any two ordered rings of type III (or of type IV) are isomorphic. Whether type V contains non-isomorphic ordered rings is an open question.

Ky Fan (Notre Dame, Ind.).

See also: Liu, p. 125; Bauer, p. 177.

Rings, Fields, Algebras

Villamayor, Orlando. Sur les équations et les systèmes linéaires dans les anneaux associatifs. C. R. Acad. Sci. Paris **240** (1955), 1681–1683.

Villamayor, Orlando. Sur les équations et les systèmes linéaires dans les anneaux associatifs. II. C.R. Acad. Sci. Paris **240** (1955), 1750–1751.

Let A be an associative ring, $W=\{x_j\}$, where $j \in J$ of cardinality β (called the indeterminates), and suppose that A and W are disjoint. The author defines a monomial to be an n-tuple $(\alpha_1, \cdots, \alpha_n)$ (n variable) where the α_i are either in A or in W and where no two successive ones come from A. He then defines equality and the negative for such monomials and similarity. Let Ω be those monomials for which at least one α_i is the 0 of A. He defines polynomials as the formal sum of monomials and then defines equality, addition and multiplication for his polynomials. This way he obtains an associative ring $\Gamma_\beta(A)$ which contains a ring isomorphic to A. If $B \supseteq A$ is a ring, then B is a homomorphic image of $\Gamma_\beta(A)$ for some appropriate β. The extensions of A are characterized as those homomorphic images of $\Gamma_\beta(A)$ in which the kernel of the homomorphism does not intersect A (as imbedded in $\Gamma_\beta(A)$). The author intends to study all extensions of A, those for which the unit element of A is also that of the extension, and commutative extensions of commutative rings. If $f(x_1, \cdots, x_n) \in \Gamma_\beta(A)$, then when studying solutions of $f(x_1, \cdots, x_n)=0$ he refers to these as A-equations. He defines algebraic elements and algebraic extensions (an element of an algebraic extension need not be algebraic) and states a necessary and sufficient condition that a system of A-equations $p_\alpha(x_{\alpha_1}, \cdots, x_{\alpha_n}, \cdots)=0$ be solvable in some extension of A. He also states that if $f(x)=0$ is solvable as an A-equation it is also solvable as an A'-equation, where A' is the ring obtained by adjoining a unit to A. In the second paper he studies necessary and sufficient conditions for the solutions of systems of linear equations, and questions of algebraic extensions, and of imbedding rings into a particular class of rings, which he characterizes. No proofs or indication of proofs are given.

I. N. Herstein (Philadelphia, Pa.).

Herstein, I. N. Two remarks on the commutativity of rings. Canad. J. Math. **7** (1955), 411–412.

In previous work the author established: i) if for each x in a ring R, $x^{n(x)}$ is in the center of R for some positive integer $n(x)$, then R is commutative or $[R, R]$ generates a nil ideal [same J. **5** (1953), 238–241; MR **14**, 719]; and a generalization of ii) if $x^{n(x)}-x$ is in the center of R for $n(x)>1$, then R is commutative [Amer. J. Math. **75** (1953), 105–111; MR **14**, 613]. In this note he proves that the respective hypotheses may be replaced by: i) for each x and y in R, $x^{n(x,y)}$ commutes with y for some integer

$n(x, y)$; and ii) $x^{n(x,y)}-x$ commutes with y. The idea is that such an assumption implies the original hypothesis for finitely generated subrings which in turn implies the global hypothesis. *W. G. Lister* (Providence, R.I.).

Nagata, Masayoshi. Basic theorems on general commutative rings. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 59–77.

Rapide exposé de théorèmes importants dans la théorie des anneaux commutatifs, relatifs plus particulièrement aux questions suivantes: idéaux premiers, anneaux de fractions, éléments entiers et extensions entières, radical de Jacobson et ses relations avec les modules, anneaux d'Artin, rang d'un idéal, anneaux noethériens intégralement clos. Comme le dit l'auteur lui même, les résultats démontrés sont, ou bien connus, ou bien des généralisations naturelles de résultats connus. Certaines démonstrations sont nouvelles, et plus simples que les démonstrations connues. Parmi elles citons celle du "going down theorem" de Cohen-Seidenberg (utilisant les automorphismes d'une extension normale), et celle du théorème de Krull disant que, si un idéal d'un anneau noethérien est engendré par r éléments, alors ses idéaux premiers isolés sont de rang $\geqq r$. *P. Samuel* (Clermont-Ferrand).

Amitsur, S. A. On rings with identities. J. London Math. Soc. **30** (1955), 464–470.

It is still not known whether a ring A satisfying a polynomial identity can be embedded in a matrix ring C_n over a commutative ring C. The author first gives an affirmative answer if A has no divisors of 0; indeed A can then be embedded in a finite-dimensional division algebra. Next it is shown that A/N can be embedded in C_n where N is the union of all nil ideals of bounded index; moreover, C_n maintains all identities satisfied by A. This strengthens an earlier result of the author [Proc. Amer. Math. Soc. **4** (1953), 27–34; MR **14**, 613] where I was the Baer lower radical of A. Application: if A is nilpotent as a Lie ring under commutation, then any commutator generates a nil ideal of bounded index. The same thing holds if A is Lie solvable provided the characteristic is not 2.
I. Kaplansky (Chicago, Ill.).

Amitsur, S. A. The T-ideals of the free ring. J. London Math. Soc. **30** (1955), 470–475.

A T-ideal in a free ring is an ideal generated by an identity; in other words it is the kernel obtained when one imposes an identity. The author's main result is that any T-ideal is primary. Stated in more direct terms, this says that if a ring satisfies an identity $fg=0$, then it satisfies $f^n=0$ or $g^n=0$. Some special cases are noted.
I. Kaplansky (Chicago, Ill.).

Szele, T. On a finiteness criterion for modules. Publ. Math. Debrecen **3** (1954), 253–256 (1955).

The author proves that if a ring satisfies both the ascending and descending chain conditions for its subrings then the ring must be finite. The proof is first given for division rings which follows from the observation that the ring then is absolutely algebraic over a prime field of characteristic $p>0$ and then invoking a result of Jacobson. The general case follows by piecing the ring modulo its radical together with matrix rings which turn out to be finite from the division-ring case. The radical is then shown to be finite and the theorem is thereby proved. *I. N. Herstein* (Philadelphia, Pa.).

Szele, T. Nilpotent Artinian rings. Publ. Math. Debrecen **4** (1955), 71–78.

An Artinian ring is one satisfying the descending chain condition on left ideals. This paper carries out a study of nilpotent Artinian rings. Using heavily the decomposition of the additive group of a nilpotent Artinian ring R, the author shows it must satisfy the descending chain condition for additive subgroups, and so, using a result of Kurosh (which the author also proves in this paper) the additive group of R is pinned down fairly completely. The problem then reduces to studying p-rings, where p is a prime, which are Artinian. Let $m>0$ be the least integer for which the group p^mR contains no non-zero elements of finite height, and let R^* be $\{x \in R | p^m x=0\}$. R^* is a finite nilpotent subring of R and is called the kernel of R. The author proves, amongst other things, the following. Let $F(R^*)$ be the nilpotent Artinian p-rings having the same kernel R^*. If R^* is any finite nilpotent p-ring, then if $R^*\neq 0$ it is the only finite member of $F(R^*)$, and every R in $F(R^*)$ has as additive subgroup a group obtainable in a well-defined manner (explicitly given in the paper). Conditions for isomorphism of two members of $F(R^*)$ are explicitly given; also, $F(R^*)$ contains only a finite number of non-isomorphic rings. $F(R^*)$ has members other than R^* if and only if R^* has an annihilator of order p^m; also the commutativity of the kernel of R implies that of R. There exist only a countable number of non-isomorphic nilpotent Artinian rings. The author adds a remark that he only needs the descending chain condition on two-sided ideals for his results to remain valid.
I. N. Herstein (Philadelphia, Pa.).

Nishi, Mieo. On the dimension of local rings. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 7–9.

Soit A un anneau local tel que, pour tout idéal premier isolé $\mathfrak{p}_i$ de (0) dans le complété $\hat{A}$, on ait $\dim(\hat{A}/\mathfrak{p}_i)=\dim(\hat{A})$ $(=\dim(A))$. Alors, pour tout idéal premier $\mathfrak{p}$ de A, on a $\dim(A/\mathfrak{p})+\dim(A_\mathfrak{p})=\dim(A)$. Ce résultat généralise et utilise un résultat de I. S. Cohen (cas où A est complet et sans diviseurs de zero); il généralise aussi un résultat de C. Chevalley (cas où A est un anneau local géométrique).
P. Samuel (Clermont-Ferrand).

Abhyankar, Shreeram, and Zariski, Oscar. Splitting of valuations in extensions of local domains. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 84–90.

Soient R un anneau local régulier de dimension >1 et d'égales caractéristiques, M son idéal maximal, K son corps des fractions, K° une extension séparable de degré fini $(\neq 1)$ de K. Il existe alors une infinité de valuations discrètes de rang 1 de K, admettant M pour centre sur R, et qui se décomposent dans K°. Au cours de la démonstration les auteurs démontrent divers lemmes relatifs à l'existence de valuations discrètes de rang 1 de centres donnés, et aux transformations quadratiques. Comme application on obtient une démonstration simple d'un théorème de Nagata relatif à la fermeture intégrale d'un anneau local intégralement clos dans un corps de décomposition de son idéal maximal.
P. Samuel (Clermont-Ferrand).

Abhyankar, Shreeram. Splitting of valuations in extensions of local domains. II. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 220–223.

Les notations sont comme ci dessus. Le résultat de l'article précédent est renforcé comme suit: R désigne un anneau local d'intégrité et de dimension >1 qui est, soit

régulier et d'égales caractéristiques (cas (a)), soit un anneau à noyau (cas (b)); il existe alors une infinité de valuations discrètes de rang 1 de K, admettant M pour centre sur R, et ayant exactement $[K^\circ:K]$ prolongements distincts à K°. En se servant du résultat de l'article précédent, des suites de transformations quadratiques, et d'une récurrence sur $[K^\circ:K]$, le théorème est d'abord démontré dans le cas (a) pour $\dim(R)=2$. Une réduction de nature simple permet alors de passer au cas général de (a). Le cas (b) est ensuite réduit à celui ci en utilisant le fait que le complété $\hat{R}$ de R n'a pas d'éléments nilpotents et est un module de type fini sur un anneau local régulier (le complété du noyau de R). La démonstration simple du théorème de Nagata est étendue au cas d'un anneau local d'intégrité à noyau. *P. Samuel* (Clermont-Ferrand).

Hua, Loo-Keng. A theorem on matrices over a sfield and its applications. J. Chinese Math. Soc. (N.S.) 1 (1951), 110–163. (Chinese summary)

La première partie de ce travail ne fait substantiellement que retrouver, en langage de matrices, un théorème de Chow [Ann. of Math. (2) **50** (1949), 32–67; MR **10**, 396] sur les transformations biunivoques de grassmanniennes conservant l'„adjacence"; les méthodes sont d'ailleurs essentiellement les mêmes, mais l'article de Chow n'est pas cité dans la bibliographie. L'auteur utilise ensuite ce résultat pour donner une caractérisation des transformations biunivoques additives $A\to A^\sigma$ d'un anneau de matrices (sur un corps commutatif ou non), telle que l'on ait $(ABA)^\sigma=A^\sigma B^\sigma A^\sigma$; il se rencontre ici avec les résultats obtenus simultanément par Jacobson et Rickart [Trans. Amer. Math. Soc. **69** (1950), 479–502; MR **12**, 387]. Enfin, la partie la plus originale de l'article concerne le problème analogue pour les transformations satisfaisant à la condition $[A,B]^\sigma=[A^\sigma,B^\sigma]$ („automorphismes de Lie"); l'auteur parvient à les déterminer complètement lorsque la caractéristique du corps des coefficients est différente de 2 ou 3, et lorsqu'il s'agit d'un anneau de matrices d'ordre >2, au moyen d'une série de lemmes très ingénieux sur les commutateurs de matrices de divers types particuliers. *J. Dieudonné* (Evanston, Ill.).

Kochendörffer, Rudolf. Zwei Reduktionssätze zum Einbettungsproblem. Wiss. Z. Univ. Rostock. Reihe Math. Nat. 2 (1953), 61–66.

Nochmalige, geringfügig abgeänderte Darstellung einer früher veröffentlichten Arbeit des Verf. [Math. Nachr. **2** (1949), 245–250; MR **11**, 314] ergänzt durch die Bemerkung, daß sich danach das Ergebnis einer inzwischen erschienenen Arbeit von E. Inaba [Nagoya Math. J. **4** (1952), 55–61; MR **14**, 139] über das Einbettungsproblem bei Normalteiler vom Typus $(p,\cdots,p)$ und bei besonderen Voraussetzungen über die übrigen Daten zu einer notwendigen und hinreichenden Existenzbedingung für eine einbettende Algebra und schärfer sogar für einen einbettenden Körper abrundet. *H. Hasse* (Zbl **52**, 271).

Albert, A. A. On Hermitian operators over the Cayley algebra. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 639–640.

Let C be the usual Cayley algebra over the field R of all real numbers, $x\to x^*$ the usual involution of C, and L the set of all n-tuples $X=(x_1,\cdots,x_n)$ with coordinates x_i in C. Then L is a vector space of dimension $8n$ over R, and the author calls a linear transformation T of L over R Hermitian in case $(XT,Y)=(X,YT)$ for all X, Y in L, where $(X,Y)=\sum_{i=1}^n x_iy_i^*$ in C. It is proved that the set A of all Hermitian operators on L is isomorphic to the special Jordan algebra of all n-rowed real symmetric matrices. *R. D. Schafer* (Storrs, Conn).

Mead, D. G. Differential ideals. Proc. Amer. Math. Soc. **6** (1955), 420–432.

The author adds to investigations by H. Levi [Trans. Amer. Math. Soc. **51** (1942), 532–568; MR **3**, 264] concerning the membership of power products in the differential ideals $[y^p]$ (p a positive integer) and $[uv]$ of $R\{y\}$ and $R\{u,v\}$ respectively (R the field of rationals). He simplifies Levi's reduction process by showing that some critical coefficients can be computed solely from the derivative subscripts without the steps of the reduction. In a special case a new sufficient condition for nonmembership is given which increases the number of power products known to be outside $[uv]$. For certain sets of power products the question of membership of its elements in $[uv]$ is shown to be reducible to proper subsets. Finally it is demonstrated by an example that a general necessary criterion for membership of a power product in $[y^p]$ must require more information than weight and degree. *A. Jaeger* (Cincinnati, Ohio).

Jaeger, Arno. A representation of multidifferential polynomials in fields of prime characteristic. Math. Ann. **130** (1955), 1–6.

Let f be a field of characteristic $p>0$, let $x_1,\cdots,x_n$ be algebraically independent over f, and let $F=f(x_1,\cdots,x_n,y)$ be a separable extension of $f(x_1,\cdots,x_n)$ of finite degree; the f-derivations of F generate a ring Ω which as an F-algebra is generated by $D_i=\partial/\partial x_i$ $(1\leq i\leq n)$, and an element of F is annihilated by every f-derivation of F if and only if it is an element of $C=f(x_1^p,\cdots,x_n^p,y^p)$. The author's theorem I asserts that $x_1^{i_1}\cdots x_n^{i_n}$ $(0\leq i_1<p,\cdots,0\leq i_n<p)$ is a basis of F/C; actually this is an immediate consequence of the elementary fact that if y is separable algebraic over a field k of characteristic $p\neq0$ then $k(y^p)=k(y)$. The author notes that the left ideal X generated by $D_1^p,\cdots,D_n^p$ in Ω is maximal two-sided, and then proves (theorem II) that Ω/X is C-isomorphic with the endomorphism ring $E(F,C)$ of the vector space F over C. *E. R. Kolchin* (New York, N.Y.).

Jaeger, Arno. A relation between adjoint multidifferential polynomials and transposed matrices for fields of prime characteristic. Math. Ann. **130** (1955), 7–10.

Notation as in the preceding review; every element A of Ω/X has a unique representative of the form

$$\sum a_{i_1\cdots i_n}D^{i_1}\cdots D^{i_n}\ (0\leq i_1<p,\cdots,0\leq i_n<p);$$

denote by A^* the element of Ω/X with the adjoint representative

$$\sum(-1)^{i_1+\cdots+i_n}D_1^{i_1}\cdots D_n^{i_n}a_{i_1\cdots i_n}.$$

If each element of $E(F,C)$ is represented by its matrix relative to the basis $(x_1^{i_1}\cdots x_n^{i_n})$, the isomorphism in theorem II above gives rise to an isomorphism μ of Ω/X onto the ring of matrices over C with indexing set $L\times L$, L being the set of n-uples $i=(i_1,\cdots,i_n)$ with $0\leq i_1<p$, $\cdots$, $0\leq i_n<p$. For any such matrix $a=(a_{ij})$ let a^+ denote the transpose of the matrix (b_{ij}), where $b_{ij}=a_{i'j'}$ and for each n-uple $i=(i_1,\cdots,i_n)$ we have let

$$i'=(p-1-i_1,\cdots,p-1-i_n).$$

Theorem: $\mu(A^*)=\mu(A)^+$.
E. R. Kolchin (New York, N.Y.).

Seidenberg, A. On separating transcendency bases for differential fields. Proc. Amer. Math. Soc. 6 (1955), 726–728.

In a previous paper [Trans. Amer. Math. Soc. 73 (1952), 174–190; MR 14, 130] the author gave, for an extension G of an ordinary differential field F, a definition of "differentially algebraic" which led to a definition of "differential transcendence basis" and of "differential transcendence degree"; he then proved, in verbal analogy with MacLane's theorem on separable field extensions, that if G/F is finitely generated (as a differential field extension) then there exists a differential transcendence basis B which has the property that G is separable over $F\langle B\rangle$. He now shows, using his definitions, that every differential transcendence basis of G/F has this property.

E. R. Kolchin (New York, N.Y.).

Raševskiĭ, P. K. The theory of spinors. Uspehi Mat. Nauk 10 (1955), no. 2(64), 3–110. (Russian)

This is a largely expository article, aimed at readers with only a modest background in algebra and analysis. Let R_n^+ be complex Euclidean (not unitary) n-dimensional space, with an orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and inner product $(\sum_{j=1}^n a_j\mathbf{e}_j, \sum_{k=1}^n b_k\mathbf{e}_k)=\sum_{j=1}^n a_jb_j$ (where the a_j and b_k are complex numbers). A concrete construction of the Clifford algebra C_n^+ of dimension 2^n is given, as follows. Basis elements of C_n^+ are the identity element 1 and the skew-symmetric contravariant tensors $\mathbf{e}_{p_1\cdots p_k}$ of rank k ($k=1, \cdots, n$; $p_1, \cdots, p_k=1, 2, \cdots, n$) whose co-ordinates $a^{i_1\cdots i_k}$ in the basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ are $+1$, -1, or 0, according as $(i_1 \cdots i_k)$ is obtainable from $(p_1 \cdots p_k)$ by an even permutation, an odd permutation, or no permutation. The product $\mathbf{e}_{p_1\cdots p_k}\mathbf{e}_{q_1\cdots q_m}=(-1)^N\mathbf{e}_{r_1\cdots r_s}$, where the r's are the indices appearing exactly once among the p's and q's, and the number N is determined by the conditions $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}=\mathbf{e}_{i_1i_2\cdots i_k}$, $\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_j\mathbf{e}_i$ if $i\neq j$, and $\mathbf{e}_i\mathbf{e}_i=1$.

The simplicity of C_n^+ is demonstrated for the case $n=2\nu$, ν being a positive integer. An explicit isomorphism is set up in this case between C_n^+ and the algebra of all linear transformations on complex affine space S_{2^ν} of dimension 2^ν. This isomorphism is constructed by use of the Clifford algebra C_ν^+, which is identified with S_{2^ν}. The representation so obtained is called the spinor representation of C_n^+, and the elements of S_{2^ν} are called spinors. The essential uniqueness of this representation is proved. Fundamental tensors in S_{2^ν} corresponding to various automorphisms and anti-automorphisms of C_n^+ are studied in enormous detail. The spinor representations of the rotation group are constructed carefully. Invariance, or the precise degree of non-invariance, of each of the tensors defined, is carefully discussed. Various real subalgebras of C_n^+ are studied. As each new concept is introduced, it is examined in detail for the cases $n=2$ and $n=4$. The Clifford algebras C_n^+ for odd n are also studied. A principal aim is to establish connections with the spinor apparatus of mathematical physics, and these connections (even to the point of discussing various authors' notations) are emphasized continually. Parts of the treatment depend on the author's "Riemannian geometry and tensor analysis" [Gostehizdat, Moscow, 1953; MR 16, 1051]. For those unfamiliar with this work and unable to make their own computations, the present paper may in places be obscure. However, being carefully written and nearly self-contained, this paper should be useful to a large group of mathematicians and physicists.

E. Hewitt.

See also: Fröhlich and Sheperdson, p. 119; Wang, p. 121; Roşculeţ, p. 145; Dieudonné, p. 174, 175; Jenner, p. 175; Cohn, p. 175.

Groups, Generalized Groups

Have

★**Kurosh, A. G. The theory of groups. Vol. I.** Translated and edited by K. A. Hirsch. Chelsea Publishing Co., New York, N.Y., 1955. 272 pp. $4.95.

Translation of Parts I and II (chapters 1–8) of the author's Teoriya grupp, 2d ed. [Gostehizdat, Moscow, 1953; MR 15, 501]. The Bibliography of the original, supplemented with some additional references, is included in this volume. The translator has added some Appendices commenting on various matters in the text. In addition, misprints have been corrected and a few minor alterations have been made.

Tsao, Shih-Hua. On groups of order $g=p^2g'$. J. Chinese Math. Soc. 2 (1953), 167–202. (Chinese. English summary)

In this paper the author investigates groups $\mathfrak{G}$ of order $g=p^2g'$, with $(p, g')=1$ and p prime. It is his aim to show that the behaviour of the characters of $\mathfrak{G}$ depends strongly on the structure of certain of its subgroups, while these subgroups can be described easily by means of groups with orders containing p only to the first power, which have been studied more closely by R. Brauer together with others. In fact, he considers p-Sylow subgroups $\mathfrak{p}$ of $\mathfrak{G}$, subgroups $\mathfrak{p}_1$ of order p in $\mathfrak{p}$, their normalizers and centralizers: $\mathfrak{N}(\mathfrak{p})$, $\mathfrak{C}(\mathfrak{p})$, $\mathfrak{N}(\mathfrak{p}_1)$ and $\mathfrak{C}(\mathfrak{p}_1)$. The work is chiefly restricted to the case of abelian $\mathfrak{p}$ of type (p, p), but the method applies to the simpler cyclic case too. $\mathfrak{C}(\mathfrak{p})$ is easily seen to be a direct product $\mathfrak{p}\times\mathfrak{W}$, $\mathfrak{W}$ being of order prime to p, and the factor-group $\mathfrak{N}(\mathfrak{p})/\mathfrak{C}(\mathfrak{p})$ of order dividing $(p-1)^2(p+1)$ can be shown to be a group of well-known type. Furthermore, $\mathfrak{C}(\mathfrak{p}_1)$ is a direct product $\mathfrak{p}_1\times\mathfrak{V}$. Between the characters of $\mathfrak{G}$ and the characters of these subgroups the following results are obtained: The blocks of defect α of $\mathfrak{G}$ are in one-to-one correspondence to the classes of irreducible characters of $\mathfrak{W}$ associated in $\mathfrak{N}(\mathfrak{p})$, and the blocks of defect 1 of $\mathfrak{G}$, each of which has a uniquely determined (within conjugation) defect group $\mathfrak{p}_1$ of order p, are in one-to-one correspondence to the classes of irreducible characters of $\mathfrak{V}$ associated in $\mathfrak{N}(\mathfrak{p}_1)$. Also, the nature of irrationalities appearing in the decomposition numbers $d_{\mu\varrho}^{t}$ of $\mathfrak{G}$ are studied as to their dependence on the structures of these normalizers and centralizers.

H.-F. Tuan (Zbl 52, 260).

Baer, Reinhold. Nilgruppen. Math. Z. 62 (1955), 402–437.

A nilgroup is a group G satisfying: (A) Every cyclic subgroup of G is accessible (nachinvariant). This condition (A) lies strictly between the conditions: (B) G is an Engel group, and (C) G is nilpotent of finite class. A fourth condition, also intermediate between (B) and (C) is: (D) G is upper nilpotent (i.e. is a term of its transfinite upper central series). In general (A) and (D) are incomparable, though in certain circumstances, for instance, if every element of G is in a finite normal subgroup, they coincide.

In any group G the generators of accessible cyclic subgroups form a characteristic subgroup $R(G)$, the unique maximal normal nilsubgroup of G. Thus a group which is generated by its accessible cyclic subgroups is a nilgroup. Furthermore, a group which is generated by a finite number of accessible cyclic subgroups is nilpotent of finite class, so that a nilgroup is locally nilpotent. In particular, finitely many elements of finite order in a

nilgroup generate a finite subgroup, so that the elements of finite order form a characteristic subgroup.

These and other theorems are applied to obtain a large number of characterisations of groups of finite class with maximal or minimal condition, and of finite soluble groups. These characterisations involve chain or finiteness conditions of various sorts, and a number of nilpotency or solubility conditions. In particular, considerable play is made with the notion of subsolubility, a group being subsoluble if every homomorph $\neq 1$ has an accessible cyclic subgroup $\neq 1$. This has the advantage over some notions of solubility, that if G has a subsoluble normal subgroup H such that G/H is subsoluble, then G is subsoluble. *Graham Higman* (Oxford).

Baer, Reinhold. Finite extensions of Abelian groups with minimum condition. Trans. Amer. Math. Soc. **79** (1955), 521–540.

If G is a group which has a centre of finite index and which satisfies the minimal condition for subgroups, then (a) G is a torsion group and every torsion group of automorphisms of G is finite, (b) classes of conjugate elements of G are finite and the minimal condition is satisfied by the abelian subgroups of G, and (c) the number of elements of prime order in any infinite homomorph of G is finite and non-zero. Conversely, each of (a), (b) and (c) implies that G has a centre of finite index and satisfies the minimal condition. There are similar, but more complicated, characterisations of groups with minimal condition and an abelian subgroup, not necessarily the centre, of finite index. As interesting by-products, it emerges that a group is finite if it is a torsion group and has only a finite number of automorphisms, or if it has only a finite number of endomorphisms. Also that maximal p-subgroups of a group with minimal condition and an abelian subgroup of finite index are conjugate.

Graham Higman (Oxford).

Itô, Noboru. Über das Produkt von zwei abelschen Gruppen. Math. Z. **62** (1955), 400–401.

Let $G=AB$ where A and B are abelian subgroups of the group G. The author shows that the commutator subgroup of G is abelian. In the finite case, he finds conditions for the existence of a proper normal subgroup which includes or which is included in A or B.

F. Haimo (St. Louis, Mo.).

Itô, Noboru. On the number of isomorphic classes of nonnormal subgroups in a finite group. Acta Sci. Math. Szeged **16** (1955), 9–11.

Let G be a finite group. If half the number $r(G)$ of isomorphic classes of non-normal subgroups of G, diminished by 1, is less than the number of distinct prime factors $\tau(G)$ in the order of G, then G is either solvable or is the icoshedral group A_5. This strengthens results of Trofimov [Mat. Sb. N.S. 33(75) (1953), 45–72; MR **15**, 286], in which moreover $\varrho(G)$ could be replaced by $r(G)$.

J. L. Brenner (Pullman, Wash.).

Łoś, Jerzy. On the complete direct sum of countable abelian groups. Publ. Math. Debrecen 3 (1954), 269–272 (1955).

A complete direct sum of countable abelian torsion-free groups is decomposable into a discrete direct sum of countable groups if and only if every summand, with at most finitely many exceptions, is a divisible group.

R. A. Good (College Park, Md.).

Zacher, Giovanni. Sugli elementi modulari in un p-gruppo. Rend. Sem. Mat. Univ. Padova **24** (1955), 165–182.

An element of a lattice is called modular if together with each modular sublattice it generates a modular sublattice. This terminology may be applied to subgroups of a group considered as elements of the lattice of all subgroups. The author proves that a modular subgroup of a finite p-group is permutable with every subgroup. Using this, he proves that a finite group G of exponent p is abelian if it has a modular subgroup $\neq 1$, G. An application concerning the Frattini subgroup of a finite p-group, and an example of a finite p-group whose lattice of subgroups is not modular, but which contains a nontrivial modular subgroup, are given.

D. G. Higman (Missoula, Mont.).

Zacher, Giovanni. Un criterio di non semplicità di un gruppo finito. Rend. Sem. Mat. Univ. Padova **24** (1955), 215–219.

See the preceding review for definitions of terms used here. In the present paper it is proved that a finite simple group G cannot contain a modular subgroup $\neq 1$, G. An essential preliminary result, the proof of which involves a theorem of Burnside to be found in Zassenhaus's Lehrbuch der Gruppentheorie [Bd. 1, Teubner, Leipzig-Berlin, 1937, p. 137] is the following: A modular subgroup H of a finite group is permutable with every subgroup by elements of order prime to the order of H.

D. G. Higman (Missoula, Mont.).

Honda, Kin'ya. On a decomposition theorem of primary groups. Comment. Math. Univ. St. Paul. **4** (1955), 53–66.

This paper contains some additional remarks on the author's decomposition theorem [same Comment. **2**, 71–83 (1954); MR **15**, 851]. This theorem runs to the effect that if G is an (additive) abelian p-group of horizontal exponent n, then G can be represented as a direct sum $A+B$, where the subgroup A of G is a direct sum of cyclic groups of order p^n, and the subgroup B of G has a horizontal exponent greater than n or is complete. The horizontal exponent of a non-complete abelian p-group G is defined to be the smallest natural number n for which there exists an element of order p^n contained in G but not contained in pG. The mean result of the present paper is the proof that the direct summands A and B in the above theorem are (up to an isomorphism) uniquely determined by G.

A. Kertész (Debrecen).

Liu, Meng-Hui. Structure of completely reducible groups and rings. J. Chinese Math. Soc. (N.S.) **1** (1951), 207–213. (Chinese summary)

The author proves that (1) an operator group is completely reducible if and only if it is a direct product of irreducible admissible normal subgroups. He states some immediate consequences of (1) and then applies (1) to a ring R whose left-ideal lattice is complemented and concludes that (2) R is its own (left) socle [J. Dieudonné, Bull. Soc. Math. France **70** (1942), 46–75; MR **6**, 144]. (Reviewer's comments: R. L. Blair [Trans. Amer. Math. Soc. **75** (1953) 136–153; MR **15**, 4] obtained (1) and (2) (among other results) independently as consequences of a general lattice-theoretic lemma. The work of Dieudonné cited by the reviewer was apparently inaccessible to the author.)

M. F. Smiley (Iowa City, Iowa).

Kostrikin, A. I. Solution of a weakened problem of Burnside for exponent 5. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 233–244. (Russian)

A maximal nil Lie-ring of exponent ϱ with k generators is obtained as a quotient ring $L_1=L/I$ where L is a free Lie-ring with generators $x_1, x_2, \cdots, x_k$ and I is the ideal generated by the elements of the form $[uv^{\varrho-1}]$. The author proves that if $\varrho=5$, L has two generators and a field of operators with characteristic$=5$, then L_1 is nilpotent and its order is $\leqq 5^{34}$. By using the relationship between periodic groups and Lie-rings [see, e.g., Sanov, same Izv. **16** (1952), 23–58; MR **13**, 721] he obtains as a consequence the solution of the weakened problem of Burnside for exponent 5: Let F be a maximal periodic group with period$=5$ with two generators, $F_\infty=\bigcap_{i=1}^\infty F_i$, where F_i is the ith term in a descending central series of F. Then the nilpotent periodic group $F_0=F/F_\infty$ is finite of order $\leqq 5^{34}$. *J. Levitzki* (Jerusalem).

Ehrenpreis, L., and Mautner, F. I. Uniformly bounded representations of groups. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 231–233.

For some time it has been conjectured that a uniformly bounded representation of a group is equivalent to a unitary representation. This paper settles the question by presenting a counter-example.

The authors consider the group of real 2 by 2 matrices of determinant 1: For any complex-valued function $f(x)$ of the real variable x, any complex number $s=u+it$, and any element g of G one puts

$$[M(g,s)f](x)=[dgx/dx]^s f(gx)=|bx+d|^{-2s}f[(ax+c)/(bx+d)],$$

where g is the matrix $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$. The representation space is the completion of the space of equivalence classes of measurable functions $f(x)$ for which

$$\|f\|_u{}^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|^{-2u}f(x)\overline{f(y)}dx\,dy$$

is finite, under the inner product

$$(f,g)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|^{-2u}f(x)\overline{g(y)}dx\,dy.$$

The example one is looking for then turns out to be the representation $g\rightarrow M(g,s)$ for s not real and $0<u<\frac{1}{2}$. *E. L. Griffin, Jr.* (Ann Arbor, Mich.).

Robinson, G. de B., and Taulbee, O. E. On the modular representations of the symmetric group. VI. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 596–598.

In a previous paper G. de B. Robinson [Canad. J. Math. **7** (1955), 391–400; MR **17**, 12] established a 1:1 correspondence between the indecomposable representations of a given p-block and the p-regular diagrams. A diagram is p-regular if it does not contain a set of p equal rows. The p-regular diagrams are called the heads in a p-block B. The associated diagrams of the conjugate representations form another p-block B'. To the heads in B' correspond diagrams in B which are called the feet. If the partitions in the block B are in dictionary order, the only members of B which contain a given indecomposable lie between the corresponding head and foot.

The authors describe a raising operator Y_p which enables one to determine the multiplicity with which each intermediate diagram contains the given indecomposable. This enables one to compute the complete D-matrix of the block which shows the manner in which the ordinary representations separate into indecomposables. The authors state that the proof of the results will be given elsewhere. *D. E. Littlewood* (Bangor).

Maak, Wilhelm. Fastperiodische Funktionen auf der Modulgruppe. Math. Scand. **3** (1955), 44–48.

In order to be able to characterize all almost periodic functions over a finite group it is sufficient to obtain a complete system of non-equivalent, irreducible, unitary representations of the group [see Maak, Fastperiodische Funktionen, Springer, Berlin, 1950; MR **13**, 29]. This characterization alone is however insufficient for the actual construction of the almost periodic functions. The problem of their construction is solved here for the particular case of the principal congruence subgroup $\Gamma(2)$ of the full modular group, and for the case of unitary representations of first degree. The free group $\Gamma(2)$ is generated by $S(\tau)=\tau+2$ and $T(\tau)=\tau/(-2\tau+1)$; if $X\in\Gamma(2)$, then $X=S^{n_1}T^{m_1}\cdots S^{n_r}T^{m_r}$. If the unitary representations of S and T are $D(S)=A$, $D(T)=B$, respectively, with $|A|=|B|=1$, then $D(X)=A^sB^t$ with $s=n_1+\cdots+n_r$, $t=m_1+\cdots+m_r$. Using results of Rademacher [J. Reine Angew. Math. **167** (1932), 312–336] on the transformation of Dedekind's η-function and on the reciprocity formula for the Dedekind sums $s(h,k)$, the author shows that $\exp\left(\frac{1}{12}\pi i\Psi_{\alpha\beta}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$ is a representation of first degree of $\Gamma(2)$. Here

$$\Psi_{\alpha\beta}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\alpha\phi(M_1)+\beta\phi(M_2)-(\alpha+\beta)\phi(M_3),$$

M_1, M_2, M_3 belong to the modular group, but not, in general, to $\Gamma(2)$, and are simply related to $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma(2)$; finally,

$$\phi(M)=\phi\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{cases} b/d \text{ if } c=0 \\ (a+d)/c-12\operatorname{sgn} cs(a,|c|) \text{ otherwise.}\end{cases}$$

From the reality of $\Psi_{\alpha\beta}$ follows that the representation is unitary and the parameters α, β can always be determined so that $\Psi_{\alpha\beta}(S)=A$, $\Psi_{\alpha\beta}(T)=B$, provided that $|A|=|B|=1$, thus proving the completeness of the system of representations. *E. Grosswald* (Philadelphia, Pa.).

★ **Ballieu, Robert. Une relation d'équivalence dans les groupoïdes et son application à une classe de demi-groupes.** IIIe Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 46–50. Fédération belge des Sociétés Scientifiques, Bruxelles.

Let D be any semigroup (Fr. "demi-groupe"). The relation E in D defined by aEb if $xa=xb$ for all x in D, is a congruence. One may consider the analogous relation E_1 in $D_1=D/E$, similarly E_2 in $D_2=D_1/E_1$, etc. A set of examples is given to show that anything can happen regarding isomorphism of each term of this series with part or all of its predecessor or successor.

Let D be a semigroup containing at least one right identity element, such that every element of D has at least one left inverse with respect to at least one right identity element. Then D/E is a group, and the semigroup $\mathcal{E}$ of all right identity elements of D is an equivalence class under E. The theorem that $D=\mathcal{E}\times(D/E)$ is proved. The author does not mention that this was first proved by Suschkewitsch [Math. Ann. **99** (1928), 30–50] for finite D, and for infinite D by the reviewer [Ann. of Math. (2) **34** (1933), 865–871]. *A. H. Clifford* (New Orleans, La.).

See also: Nöbauer, p. 131; Dieudonné, p. 174, 175.

THEORY OF NUMBERS

Have

★**Scholz Arnold. Einführung in die Zahlentheorie.** Überarbeitet und herausgegeben von Bruno Schoeneberg. Sammlung Göschen Bd. 1131. Walter de Gruyter & Co., Berlin, 1955. 128 pp. DM 2.40.

This excellent introduction to the theory of numbers was first published in 1939 and was reprinted without change in 1945 [see MR **11**, 159]. No account is given of the changes made in the present edition, but a comparison shows that they are quite appreciable. The final chapter on algorithmic calculation has been omitted, and the remaining contents have been rearranged and often rewritten. References to recent work have been introduced where appropriate. The exposition is now a little fuller, but perhaps the book has lost something of its distinctive character. Occasionally the changes have been for the worse, e.g. in the proof (p. 75) of the insolubility of $x^4+y^4=z^2$, the addition of the one word "etwa" has introduced an error which was not present before.

H. Davenport (London).

Lemaître, G. Pourquoi de nouveaux chiffres? Rev. Questions Sci. (5) **16** (1955), 379–398.

The author discusses the shortcomings of the decimal system and even the Arabic digits. Certain advantages of binary system are stressed such as the fact that it takes the fun out of playing Nim. No specific alternatives are insisted upon however.

D. H. Lehmer (Berkeley, Calif.).

Booth, Andrew D. Some curiosities of binary arithmetic useful in testing binary computers. Computers and Automation **4** (1955), no. 8, 33, 42.

van der Blij, F. Gaussian sums. Euclides, Groningen **30** (1954/55), 293–298. (Dutch)

Cavallaro, Vincenzo G. Dalle identità aritmetiche di Gergonne, di Cauchy e di Lamé all'elisse di Lemoine. Giorn. Mat. Battaglini (5) **3(83)** (1955), 69–75.

Pompeiu, D. Correspondance bi-univoque entre ensembles dont les éléments sont des nombres entiers. Com. Acad. R. P. Romîne **1** (1951), 739–740. (Romanian. Russian and French summaries)

An elementary example of one-to-one correspondence between two sets of integers is presented; it permits the reconstruction of a multiplication with missing digits.

E. Grosswald (Philadelphia, Pa.).

Good, I. J. Conjectures concerning the Mersenne numbers. Math. Tables Aids Comput. **9** (1955), 120–121.

Storchi, Edoardo. Alcuni criteri di divisibilità per i numeri di Mersenne e il carattere 6^{co}, 12mo, 24mo, 48mo, dell'intero 2. Boll. Un. Mat. Ital. (3) **10** (1955), 363–375.

The author combines known criteria for 2 to be a kth power residue of a prime $p=kx+1$ for $k=2, 3, 4, 8, 16$ to obtain equivalent theorems for $k=6, 12, 24$, and 48 expressed in terms of Mersenne numbers and their divisors. Unfortunately no new results are obtained. Since a criterion is known for $k=5$ [E. Lehmer, Duke Math. J. **18** (1951), 11–18; MR **12**, 677] the author might have extended his results to $k=240$.

D. H. Lehmer (Berkeley, Calif.).

Ward, Morgan. The intrinsic divisors of Lehmer numbers. Ann. of Math. (2) **62** (1955), 230–236.

Let L and M be integers, with L and $L-4M$ positive and $M\neq 0$. Then the zeros α and β of the polynomial $f(z)=z^2-L^{\frac{1}{2}}z+M$ are real. The Lehmer numbers P_n associated with $f(z)$ are defined as follows:

$$P_n=(\alpha^n-\beta^n)/(\alpha-\beta) \text{ for } n \text{ odd},$$

and

$$P_n=(\alpha^n-\beta^n)/(\alpha^2-\beta^2) \text{ for } n \text{ even}.$$

Thus $(P)=\{P_n\}$ is a sequence of integers; it is called a real Lehmer sequence if α and β are real. A prime p is an intrinsic divisor of P_k if $p|P_k$ but $p\nmid P_n$ for index n smaller than k. (P) is said to be exceptional if some P_k with $k>2$ has no intrinsic divisor. It is shown that there are only three exceptional real Lehmer sequences; their associated polynomials are z^2-z-1, $z^2-5^{\frac{1}{2}}z+1$ and z^2-3z+2. The exceptional indices are also determined; they are all smaller than eighteen.

W. J. LeVeque.

Carlitz, L. The coefficients of certain power series. Monatsh. Math. **59** (1955), 188–193.

Let $P=\prod_0^n (1-x_i)$, $S=\sum_0^n x_i/(1-x_i)$, $\Phi=P(n+1+S)$, and for each integer $r\geqq 0$ define integers $a_{k_0k_1\cdots k_n}$ by the equation

$$((n+1)/\Phi)^{r+1}=\sum a_{k_0k_1\cdots k_n}z_0^{k_0}z_1^{k_1}\cdots z_n^{k_n},$$

where the summation is over all $k_0, k_1, \cdots, k_n$ from 0 to ∞. The author obtains various divisibility properties of the numbers $a_k=a_{kk_1\cdots k_n}$. For instance, let p be a prime $>\max(r, k_1, \cdots, k_n)$ and assume $\lambda\geqq 1$, $t\geqq 1$. Then

$$\sum_{i=0}^{\lambda}(-1)^{\lambda-i}\binom{\lambda}{i}a_{k+ip^{t}(p-1)}\equiv 0 \pmod{p^{\lambda t}}$$

for $k\geqq k_1+\cdots+k_n$ provided $p>n$, and for

$$k\geqq k_1+\cdots+k_n+\lambda t$$

otherwise. There are obvious misprints in formulas (1.3) and (3.2).

T. M. Apostol (Pasadena, Calif.).

Cugiani, Marco. Nuovi risultati sulle "catene" di numeri primi. Ann. Mat. Pura Appl. (4) **38** (1955), 309–320.

Extending the investigations of an earlier paper [same Ann. (4) **36** (1954), 121–132; MR **15**, 935] the author again considers "chains" consisting of consecutive primes whose differences satisfy given conditions. He is principally concerned with those chains in the interval $((1-\eta)\xi, \xi)$ for which the difference of two consecutive elements is always less (or those in which it is always not less) in absolute value than an expression of the form $\alpha \log \xi$, where α is a constant ($\alpha>1$ or $\alpha<1$). A number of inequalities are derived. These are related either to the extension of the maximal chain or to the total number of chains in the interval.

A. L. Whiteman.

Chowla, S., and Briggs, W. E. On the number of positive integers $\leqq x$ all of whose prime factors are $\leqq y$. Proc. Amer. Math. Soc. **6** (1955), 558–562.

Recently de Bruijn [Nederl. Akad. Wetensch. Proc. **53**, 803–812=Indag. Math. **12** (1950), 247–256; MR **12**, 11] estimated the functions $f(x, y)$ and $g(x, y)$, where $f(x, y)$ denotes the number of positive integers $\leqq x$, all of whose prime factors are $\leqq y$, and where $g(x, y)$ denotes the number of positive integers $\leqq x$, all of whose prime factors are $>y$. An interesting special case is the estimate

$f[x, (\log x)^h]=O(x^{1-1/h+\epsilon})$, where $h>2$. The present authors give a new proof of this formula and a corresponding formula for $g[x, (\log x)^h]$. Although they follow de Bruijn very closely their analysis is considerably simpler.
A. L. Whiteman (Los Angeles, Calif.).

de Bruijn, N. G. Some classes of integer-valued functions. Nederl. Akad. Wetensch. Proc. Ser. A. **58**= Indag. Math. **17** (1955), 363–367.

An integer-valued function $f(x)$ is called universal if it has the property that

$$(*)\qquad f(x+m)-f(x)\equiv 0 \pmod m$$

for all $x=0, 1, 2, \cdots$ and $m=1, 2, 3, \cdots$. It is proved that $f(x)$ is universal if and only if it has the form

$$f(x)=c_0+s_1c_1\binom{x}{1}+s_2c_2\binom{x}{2}+\cdots,$$

where the c's are integers and s_k is the least common multiple of $1, 2, \cdots, k$. Call $f(x)$ modular if (*) holds for all m without the restriction $x\geqq 0$. It is shown by an example that modular extension of a universal function is not always possible. Also it is proved that $f(x)$ is modular if and only if

$$f(x)=c_0+\sum_1^\infty s_kc_k\binom{x+[k/2]}{k},$$

where the c's are integers. Finally it is proved that a function $f(x_1, \cdots, x_n)$ is universal if and only if in the representation

$$f(x_1, \cdots, x_n)=\sum_0^\infty b(k_1, \cdots, k_n)\binom{x_1}{k_1}\cdots\binom{x_n}{k_n},$$

each $b(k_1, \cdots, k_n)$ is a multiple of s_t, where

$$t=\max(k_1, \cdots, k_n).$$

The author wishes to point out that the results of the present paper are closely related to Theorems 1 and 2 of E. G. Straus, Duke Math. J. **19** (1952), 379–395 [MR **14**, 21]. In this connection see also L. Rédei and T. Szele, Acta Math. **79** (1947), 291–320 [MR **9**, 407]. *L. Carlitz.*

Vrănceanu, G. Sur une équation arithmétique. Com. Acad. R. P. Române **3** (1953), 5–8. (Romanian. Russian and French summaries)

Let N^2 have (in decimal notation) as last digits precisely the n digits of N. Then N satisfies the diophantine equation (*) $N^2-N=A\cdot 10^n$ with some integral A. For every integer n, Pompeiu [Acad. Repub. Pop. Române. Bul.Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 1–5; MR **15**, 602; see also Dickson, History of the theory of numbers, v.1, pp. 458–459] has shown that there are two solutions N and N', satisfying $N+N'=10^n+1$. The author gives a proof of these statements, generalizes (*) to (**) $N^2-kN=A\cdot 10^n$ and obtains the result that (**) with $k=\pm 1, \pm 3$ contains only two lattice points (N, A) with positive coordinates less than 10^{n+1} *E. Grosswald.*

Grammel, R. Diophantische Vektorgleichungen. Österreich. Ing.-Arch. **9** (1955), 126–147.

Greek letters will denote so-called vector integers, i.e., 3-dimensional vectors with integers for components. Given non-null vector integers α and β there exists ϱ such that $\alpha\times\varrho=\beta$ if and only if the g.c.d. of the components of α divides the g.c.d. of the components of β. Results are also obtained for other vector equations such as $\alpha\cdot\varrho=b$, $\alpha\times\beta\times\varrho=\gamma$, $\alpha\cdot(\beta\times\varrho)+\gamma\cdot\varrho=d$, where Roman letters denote scalar integers. Some systems of vector equations are discussed. In each case the problem reduces to a set of Diophantine equations, and so is tractable by classical methods. *I. Niven* (Berkeley, Calif.).

Watson, G. L. Representation of integers by indefinite quadratic forms. Mathematika **2** (1955), 32–38.

The author considers the Diophantine equation

$$(1)\qquad f(x_1, x_2, \cdots, x_k)=n \quad (n\neq 0),$$

where f is a quadratic form with integer coefficients and where the solution is subject to the restriction

$$(2)\qquad (x_1, x_2, \cdots, x_k)\equiv(t_1, t_2, \cdots, t_k) \pmod q$$

for q positive, the t_i given and 1 the g.c.d. of the x_i. He proves the following theorem: If f is an indefinite form in four or more variables, $n\neq 0$, and the congruence $f\equiv n \pmod m$ has for every m a solution satisfying (2), then (1) has a solution satisfying (2). In particular, taking $q=1$, an indefinite form f in four or more variables represents properly every integer $n\neq 0$ for which the congruence $f\equiv n \pmod m$ is solvable for every m.

The author indicates that modifications of his method can be made for $k\geqq 5$ if $n=0$ or if n is replaced by a form g of non-zero determinant and having no more than $k-3$ variables. *B. W. Jones* (Boulder, Colo.).

Oppenheim, A., and Barnes, E. S. The non-negative values of a ternary quadratic form. J. London Math. Soc. **30** (1955), 429–439.

In connection with recent results of Barnes [Proc. London Math. Soc. (3) **5** (1955), 185–196; MR **16**, 1002] the following theorem is proved. Let $Q(x, y, z)$ be a quadratic form of determinant $d\neq 0$ and signature -1. Then there exist integers x, y, z, not all zero, satisfying

$$0\leqq Q(x, y, z)\leqq(16d/5)^{\frac{1}{3}}.$$

The second sign of equality is necessary if and only if Q is equivalent to a positive multiple of the special form $-x^2-xy-y^2+90z^2$. *J. F. Koksma* (Amsterdam).

Jones, Burton W., and Marsh, Donald. A proof of a theorem of Meyer on indefinite ternary quadratic forms. Amer. J. Math. **77** (1955), 513–525.

A proof of Meyer's theorem that a very wide set of genera of indefinite ternary forms have only one class. The proof is on the same lines, but is claimed to be more perspicuous and accurate than Meyer's paper [J. Reine Angew. Math. **108** (1891), 125–139] or the versions of Bachmann [Die Arithmetik der quadratischen Formen, Abt. 1, Teubner, Leipzig, 1898, pp. 233–251] and Dickson [Studies in the theory of numbers, Univ. of Chicago Press, 1930, pp. 35–60].
J. W. S. Cassels (Cambridge, England).

Volkmann, Bodo. Über die Klasse der Summenmengen. Arch. Math. **6** (1955), 200–207.

For a set $\mathfrak{A}$ of integers a_i, let $\Gamma(\mathfrak{A})=\sum 2^{-a_i}$. For a class A of sets $\mathfrak{A}$, let $\Gamma(A)$ be the set of numbers $\Gamma(\mathfrak{A})$. For two sets $\mathfrak{A}$, $\mathfrak{B}$, let $\mathfrak{A}+\cdot\mathfrak{B}$ be the Schnirelmann sum, consisting of all integers a_i+b_j. For two classes A, B of sets $\mathfrak{A}$, $\mathfrak{B}$ (of integers), let $A\cdot+\cdot B$ be the class of sets $\mathfrak{A}+\cdot\mathfrak{B}$. Let $\mathfrak{A}(n)$ be the number of members a of $\mathfrak{A}$ in $a\leqq n$. Let $K(\alpha)$ be the class of sets $\mathfrak{A}$ for which $\liminf \mathfrak{A}(n)/n\geqq\alpha$, and $S(\alpha)$ the class of *sets* $\mathfrak{A}+\cdot\mathfrak{B}$ for which

$$\liminf\,[\mathfrak{A}(n)+\mathfrak{B}(n)]/n\leqq\alpha.$$

Let T be the class of sets $\mathfrak{A}$ with at least two members, F

the subclass of finite sets in T, I the class of infinite sets $\mathfrak{A}$. Finally let $d(\alpha)=-\alpha\log_2\alpha-(1-\alpha)\log_2(1-\alpha)$. Then we have the following theorems. Th. 1:

$$\dim\Gamma\{K(\alpha)\cdot+\cdot K(\beta)\}=0$$

if $\alpha+\beta\geqq 1$, and is $\leqq d(\alpha+\beta)$ if $\frac{1}{2}\leqq\alpha+\beta<1$. Th. 2: $\dim\Gamma(S(\alpha))\leqq 2d(\frac{1}{2}\alpha)$. Th. 3: $\Gamma(I\cdot+\cdot I)$ is of Lebesgue measure zero. Th. 4: If $\mathfrak{A}$, $\mathfrak{B}\in I$, then the sequence $2^n\Gamma(\mathfrak{A}+\cdot\mathfrak{B})$ is non-uniformly distributed (mod 1) except (possibly) when $\lim\mathfrak{A}(n)/n=\lim\mathfrak{B}(n)/n=0$.

It is also shown that $\Gamma(T\cdot+\cdot F)$ is of Lebesgue measure zero and of Hausdorff dimension $>0{\cdot}8$; but the dimensions of $\Gamma(T\cdot+\cdot F)$, $\Gamma(I\cdot+\cdot I)$ are still undetermined.

H. D. Ursell (Leeds).

Analytic Number Theory

Meinardus, Günter. Über die Kroneckersche Grenzformel. Math. Z. 62 (1955), 347–351.

Let $\tau=x+iy$ with $y>0$ and for $\mathrm{Re}(s)>1$ set $\zeta(s)\sum_{n=1}^{\infty}n^{-s}$ and

$$Z(\tau,s)=\sum_{\substack{n,m\\ n^2+m^2\neq 0}}|n+m\tau|^{-2s}.$$

Then Kronecker's [S.-B. Preuss. Akad. Wiss. **1885,** 761–784, p. 775; **1889,** 123–135, p. 135] formula

$$(*)\quad \lim_{s\to 1}\{Z(\tau,s)-\pi/y(s-1)\}=$$
$$-\frac{2\pi}{y}\log\{\eta(\tau)\eta(-\bar\tau)\}-\frac{2\pi}{y}\log y+\frac{2\pi}{y}(\gamma-\log 2)$$

holds; here γ is the Euler constant and $\eta(\tau)$ the Dedekind function. Koecher [Arch. Math. **4** (1953), 316–321; MR **15,** 507] showed that any proof of $(**)$ $(-i\tau)^{\frac{1}{2}}\eta(\tau)=\eta(-1/\tau)$ can be adapted to yield a proof of $(*)$. Siegel [Mathematika **1** (1954), 4; MR **16,** 16] gave a simple new proof of $(**)$ and the present paper gives the corresponding proof of $(*)$. For non-integral w one has

$$(1)\qquad \pi\operatorname{cotg}\pi w=\frac{1}{w}+\sum_{m=1}^{\infty}\left\{\frac{1}{w+m}+\frac{1}{w-m}\right\},$$

and, for $y>0$, $\mathrm{Re}(s)>\frac{1}{2}$,

$$(2)\qquad \int_{-\infty}^{\infty}|w+iy|^{-2s}dw=\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-2s}$$

holds. If W_N stands for the contour of integration formed by the rectilinear segments between the 4 points $N+\frac{1}{2}\pm i\beta$, $\alpha\pm i\beta$, $0<\alpha<1$, $0<\beta<y/2$, N natural integer, then, by (1) and the theorem on residues,

$$\frac{1}{i}\int_{W_N}\{(w+m\tau)(w+m\bar\tau)\}^{-s}\operatorname{ctg}\pi w\,dw=2\sum_{n=1}^{N}|n+m\tau|^{-2s}.$$

Summing both sides over m, using repeatedly the theorem on residues, the uniformity of convergence of the sum under the integral sign on the left and (2), finally, letting N tend to infinity, one obtains the following: Let

$$K(\tau,s)=Z(\tau,s)-2\pi^{\frac{1}{2}}y^{-2s}\Gamma(s-\tfrac{1}{2})\zeta(2s-1)/\Gamma(s);$$

then $K(\tau,1)=-2\pi y^{-1}\log\{\eta(\tau)\eta(-\bar\tau)\}$. This, however, is equivalent to $(*)$, as $\Gamma'(1)=-\gamma$, $\Gamma'(\frac{1}{2})=-\pi^{\frac{1}{2}}(\gamma+2\log 2)$ and $\lim_{s\to 1}(\zeta(s)-1/(s-1))=\gamma$.

E. Grosswald.

Petersson, Hans. Über Modulfunktionen und Partitionenprobleme. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Nat. **1954**, no. 2, 59 pp.

The author considers a very general partition problem. Let $\mathfrak{S}_b$ be the set of integers congruent to $\pm b$ mod N, $b=1, 2, \cdots, N/2$, N even. Let $k_{b\nu}$, $0\leqq\nu\leqq f_b$, be even non-negative integers; and $l_{b\nu}$, $1\leqq\nu\leqq f_b$, integers $\geqq 2$. Let $p_n([k],[l],N)$ be the number of partitions of n into positive summands; if a summand lies in $\mathfrak{S}_b$ it is associated with one of the $k_{b\nu}$ "colors", and in the first k_{b0} colors it may occur arbitrarily often, in the next k_{b1} colors it occurs at most $l_{b1}-1$ times, etc. The generating function is

$$\sum_{n=0}^{\infty}p_n([k],[l],N)x^n=\prod_{b=1}^{N/2}\prod_{m=\pm b(N)}\{(1-x^m)^{-k_{b0}}$$
$$\prod_{j=1}^{f_b}(1+x^m+\cdots+x^{(l_{bj}-1)m})^{k_{bj}}\},|x|<1.$$

For this partition function the author obtains a convergent series representation of the type of the Rademacher series, as well as an asymptotic formula. The method used is not the Hardy circle method, but the author's own function-theoretic method, which he has expounded in a series of papers.

He treats also cases in which the summands satisfy additional restrictions depending on the Kronecker symbol of a real quadratic field, as well as the asymptotic value of the quotient of two partition functions in which the summands fulfill related conditions.

J. Lehner.

Petersson, Hans. Über automorphe Formen mit Singularitäten im Diskontinuitätsgebiet. Math. Ann. **129** (1955), 370–390.

Let Γ stand for a Fuchsian group of the first kind, having $\mathfrak{F}$ as one fundamental domain and the real axis as principal circle. Let K be the class of forms $\{\Gamma,-r,v\}$ defined in the upper half-plane of the variable τ, automorphic under Γ, of negative dimension $-r<-2$ and of multipliers v, $|v|=1$; and let $\hat K$ be the complementary class $\{\Gamma,r-2,v^{-1}\}$ of positive dimension $r-2$. As there are no entire automorphic forms of positive dimension (except $F\equiv 0$), the poles and principal parts of a form of $\hat K$ define it uniquely. As a consequence of the theorem on residues and the Riemann-Roch theorem, these poles and principal parts cannot be prescribed arbitrarily, but have to satisfy a "principal-parts condition". If that is satisfied, however, there exists a uniquely determined form F of $\hat K$, satisfying all prescribed conditions. The author indicates an explicit construction of F in the particular case of the modular group, actually valid for any group Γ of the first kind, provided that $\mathfrak{F}$ has a single cusp [S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950,** 417–494; MR **12,** 806]. Generalizing these results to the case of arbitrary Fuchsian groups of the first kind, with $\mathfrak{F}$ having any finite number of cusps, the author obtained for F expansions centered at the cusps, involving functions of two complex variables [Math. Ann. **127** (1954), 33–81, cited in what follows as AO; MR **15,** 686]. For each cusp, a different respresentation holds. The analytical apparatus developed in AO did not permit the construction of similar expansions for $F\in\hat K$, around an arbitrary interior point of $\mathfrak{F}$. This the author succeeds in doing in the present paper, using functions of three complex variables. The method follows in the main that of AO. As building blocks are used: (i) functions $P_m(z,s)$, the Fourier coefficients in the expansions of certain generalized Poincaré series $Q(\tau,z,s)$, which will characterize the behaviour of the form at the cusps; and (ii) functions $V_n(z,s,\omega_k)$, characterizing the behaviour at the corners ω_k of F and $V_n(z,s,c_\nu)$, characterizing the behaviour at the poles c_ν. In themselves, these functions are not automorphic forms. But if $\Lambda(z,s)$ is a linear combination of them, $\Lambda(z,s)$ becomes auto-

morphic provided that the coefficients of the linear combination are selected so that the "principal-parts condition" is fulfilled. Accordingly, the main result may be stated as follows: If $\mathfrak{F}$, K, cusps and poles with principal parts are given, such that the "principal-parts condition" holds, then there exists exactly one automorphic form $F(z)$ of class $\hat{K}$ with the properties: $F(z)$ is regular in $\mathfrak{F}$, except at the cusps, the corners ω_k and the poles c_ν; at the cusps, corners and poles, $F(z)$ has principal parts of a specified form. For every interior point s of $\mathfrak{F}$, $F(z)$ is explicitly represented by $\Lambda(z, s)$. *E. Grosswald.*

See also: Maak, p. 126.

Algebraic Number Theory

Dalen, Kåre. On a theorem of Stickelberger. Math. Scand. 3 (1955), 124–126.

Let Ω be a perfect field with respect to the valuation w. Let I denote the set of numbers of Ω such that $w(a)\geqq 0$ and the numbers such that $w(a)>0$. Then $P=I/\mathfrak{P}$ is a field; it is assumed that P is isomorphic either to the GF(q) or the field of real numbers. Using the Hensel Lemma [see, e.g., B. L. van der Waerden, Moderne Algebra, Teil I, 3rd ed., Springer, Berlin, 1950, p. 263; MR **12**, 236], the following theorem is proved. Let

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n$$

be a polynomial of $I[x]$ with discriminant $\not\equiv 0$ ($\mathfrak{P}$), and let s be the number of irreducible factors of $f(x)$ in $P[x]$. Then $(D/\Omega)=(-1)^{n-s}$, where $(D/\Omega)=+1$ or -1 according as x^2-D is irreducible or not in $\Omega[x]$. In particular, this result includes a theorem of Stickelberger [Verh. 1. Internat. Math.-Kongresses, Zürich, 1897, Teubner, Leipzig, 1898, pp. 182–193]. *L. Carlitz.*

Moisil, Gr. C. Une définition des nombres idéaux dans la théorie locale de Zolotarev. Com. Acad. R. P. Romîne **5** (1955), 279–283. (Romanian. Russian and French summaries)

In order to give a concrete content (as sets of elements of a domain of integrity $\mathfrak{O}$) to the concept of ideal number of Kummer, the author introduces an equivalence relation among the elements of $\mathfrak{O}$ as follows. If α, $\beta \in \mathfrak{O}$, then α is said to be divisible by β with respect to the rational integer m, if there exists another rational integer c, $(c, m)=1$, and such that $c\alpha$ is divisible by β in $\mathfrak{O}$. If α divides β and β divides α with respect to the same integer m, then α and β are said to be associate with respect to m. This relation is an equivalence relation and the "ideals" of $\mathfrak{O}$ are defined as the equivalence classes in $\mathfrak{O}$. Elementary properties of these "ideals" are verified and the reader is referred to Zolotareff's paper [J. Math. Pures Appl. (3) **6** (1880), 51–84, 129–166] for the theory of factorization into prime ideals. *E. Grosswald.*

Carlitz, L., and Hodges, John H. Representations by Hermitian forms in a finite field. Duke Math. J. **22** (1955), 393–405.

Let $q=p^n$, $p>2$, and suppose that $\theta \in \mathrm{GF}(q^2)$, $\theta^2=v \in \mathrm{GF}(q)$ but $\theta \notin \mathrm{GF}(q)$. Then $\alpha=a+b\theta \in \mathrm{GF}(q^2)$ if $a, b \in \mathrm{GF}(q)$. The conjugate of α is $\bar\alpha=a-b\theta$. If $A=(\alpha_{ij})$ is a square matrix, $\alpha_{ij} \in \mathrm{GF}(q^2)$, let $A^*=\bar{A}'=(\bar\alpha_{ij})'$, where the prime denotes transpose. Then A is said to be Hermitian if and only if $A^*=A$. In this paper the authors derive formulas for determining the number $N_t(A, B)$ of $m\times t$ matrices U such that $U^*AU=B$, where A and B are Hermitian, A is non-singular of order m, B is of order t and rank $r\leqq m$, and all matrices have elements in GF(q^2). Incidentally they obtain a formula for the number $N(m, r)$ of Hermitian matrices of order m and rank r. The authors also evaluate explicitly a certain sum $H(B, s)$ (useful in partition problems) which for $B=0$ reduces to $N(t, s)$. The methods employed in this paper are similar to those used previously by the first author [same J. **21** (1954), 123–137; Arch. Math. **5** (1954), 19–31; MR **15**, 604, 777] in the treatment of the analogous problems for symmetric and skew-symmetric matrices, respectively.
A. L. Whiteman (Los Angeles, Calif.).

Butler, M. C. R. The irreducible factors of $f(x^m)$ over a finite field. J. London Math. Soc. **30** (1955), 480–482.

Let $f(x)$ be an irreducible polynomial of degree n over GF(q) whose roots have (necessarily equal) multiplicative order t. Then (1) the roots of $f(x^m)$ have multiplicative order of the form $s=mt/k$ with k prime to t; (2) the number of roots of order s is $n\varphi(s)/\varphi(t)$; (3) $f(x^m)$ factors into a product of polynomials f_k, one for each divisor of m prime to t (hence one for each $s=mt/k$) and each f_k is the product of $n\varphi(s)/\varphi(t)M(s)$ irreducible factors, where $M(s)$ denotes the multiplicative order of q modulo s and φ is the Euler function.

[Reviewer's note: (1) and (2) are equivalent to statements about the multiplicative order and number of solutions of $x^m=a$ (take a to be a root of $f(x)$) and, when so phrased, are true in any cyclic group.] (3) is a result of the fact that in a finite field, an element of multiplicative order s is the root of an irreducible polynomial of degree $M(s)$. This review's notation is slightly different from the author's. *D. Zelinsky* (Kyoto).

Carlitz, Leonard. On the number of distinct values of a polynomial with coefficients in a finite field. Proc. Japan Acad. **31** (1955), 119–120.

Let GF(q) denote the finite field of order $q=p^\nu$ and put $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x$ ($a_i \in \mathrm{GF}(q)$), where $1<n<p$. Let $V=V(f)$ denote the number of distinct values of $f(x)$, $x \in \mathrm{GF}(q)$. Under the assumption that

$$f^*(u, v)=(f(u)-f(v))/(u-v)$$

is absolutely irreducible Uchiyama [same Proc. **30** (1955), 930–933; MR **16**, 906] has proved the deep result that $V>q/2$ for all $n\geqq 4$. In this note the author proves in a simple manner that $V>q/2$ on the average.
A. L. Whiteman (Los Angeles, Calif.).

Uchiyama, Saburô. Note on the mean value of $V(f)$. Proc. Japan Acad. **31** (1955), 199–201.

Let $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x$, $a_i \in \mathrm{GF}(q)$, where $1<n<p$, and let $V(f)$ denote the number of distinct values $f(x)$, $x \in \mathrm{GF}(q)$. The reviewer has proved (in the paper reviewed above) that

$$\sum_{a_i \in \mathrm{GF}(q)} V(f)\geqq \frac{q^3}{2q-1}>\frac{q^2}{2},$$

so that $V(f)>q/2$ on the average. This result may be compared with an earlier theorem of the present author [same Proc. **30** (1954), 930–933; MR **16**, 906]. In the present paper it is proved that

$$(*)\qquad \sum_{\deg f=n} V(f)=\sum_{r=1}^{n}(-1)^{r-1}\binom{q}{r}q^{n-r}\quad (1\leqq n<p),$$

where the summation on the left is over all primary polynomials of degree n without constant term. The writer remarks that (*) implies that

$$(**)\qquad V(f)=c_nq+O(1)$$

on the average, where $c_n=\sum_1^n (-1)^{r-1}/r!$, and that, for $n\leqq 4$, (**) is an accurate estimate for $V(f)$. Thus it would be of interest to know when (**) holds for a given polynomial of higher degree. It is conjectured that the absolute irreducibility of $f^*(u,v)=(f(u)-f(v))/(u-v)$ would yield this result. *L. Carlitz* (Durham, N.C.).

Uchiyama, Saburô. Sur les polynômes irréductibles dans un corps fini. II. Proc. Japan Acad. **31** (1955), 267–269.

The present paper is a continuation of an earlier one by the same author [same Proc. **30** (1954), 523–527; MR **16**, 570]. Extending Theorem 2 of the former paper, the author now obtains the following result. Let $\pi_q(m; r, t)$ denote the number of polynomials $x^m+c_1x^{m-1}+\cdots+c_m$, $c_i \in \mathrm{GF}(q)$, with assigned coefficients $c_1, \cdots, c_r; c_{m-t+1}, \cdots, c_m$, that are irreducible in $\mathrm{GF}[q, x]$. Then if $r+t\geqq 2$ and $p>\max(r, t-1)$

$$\pi_q(m; r, t)=\frac{1}{m}q^{m-r-t}+O(q^{\theta m})\quad (m\to\infty),$$

where θ $(\frac{1}{2}\leqq\theta<1)$ is a constant independent of q and m. The proof depends on the properties of certain L-functions. *L. Carlitz*, (Durham, N.C.).

Nöbauer, Wilfried. Gruppen von Restpolynomidealrestklassen nach Primzahlpotenzen. Monatsh. Math. **59** (1955), 194–202.

Let $\mathfrak{A}$ denote the ideal consisting of all residual polynomials (mod n) and Γ the ring of rational integers; then the element of $\Gamma[x]/\mathfrak{A}$ constitute a semigroup $\mathfrak{H}_n$ with respect to the operation $f(g)$, where $f, g \in \Gamma[x]$. The elements of $\mathfrak{H}_n$ with inverses constitute a group $\mathfrak{G}_n$. In a previous paper [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **162** (1953), 207–233; MR **15**, 856] it was proved that if $n=ab$, where $(a, b)=1$, then $\mathfrak{G}_n$ is isomorphic to the direct product of $\mathfrak{G}_a$ and $\mathfrak{G}_b$; also, for p prime, $\mathfrak{G}_p$ is the symmetric group $\mathfrak{S}_p$. In the present paper the group $\mathfrak{G}_{p^e}$ is studied for $e>1$ and it is shown that this group is isomorphic to the symmetry $\sum_p (\mathfrak{T}_{p^{e-1}})$, where $\mathfrak{T}_{p^{e-1}}$ is a certain subgroup of $\mathfrak{G}_{p^{e-1}}$ [see W. Specht, Schr. Math. Sem. Inst. Angew. Math. Univ. Berlin **1** (1932), 1–32; Math. Z. **37** (1933), 321–341; O. Ore, Trans. Amer. Math. Soc. **51** (1942), 15–64; MR **3**, 197]. Some properties of $\mathfrak{T}_{p^e}$ are obtained; in particular, an explicit formula for its order. The groups $\mathfrak{T}_{p^e}$ are solvable for all $e\geqq 1$. *L. Carlitz* (Durham, N.C.).

Dénes, Péter. Über Grundeinheitssysteme der irregulären Kreiskörper von besonderen Kongruenzeigenschaften. Publ. Math. Debrecen **3** (1954), 195–204 (1955).

Let p denote an odd prime, r a primitive root (mod p), $q=(p-3)/2$, ζ a primitive pth root of unity, $\Omega(\zeta)$ the cyclotomic field, $\lambda=(1-\zeta)$, $L=(\lambda)$. In a previous paper [same Publ. **3** (1954), 17–23; MR **15**, 686], the writer proved the existence of a set of independent units $\eta_1, \cdots, \eta_q$ of $\Omega(\zeta)$ such that

$$\eta_i\equiv 1+\lambda^{2e_i} \pmod{L^{2e_i+1}}\quad (i=1, \cdots, q),$$

where $2e_i=u_i(p-1)+2i$, and $u_1, \cdots, u_q$ denote the p-character of the Bernoulli numbers. Theorem 1 of the present paper guarantees the existence of a fundamental set of units of $\Omega(\zeta)$ such that

$$(*)\qquad \delta_i\equiv a_i+b_i\lambda^{2e_i'} \pmod{L^{2e_i'+1}}\quad (i=1, \cdots, q),$$

where a_i and b_i are integers prime to p and

$$2e_i'=u_i'(p-1)+2i.$$

The numbers $u_1', \cdots, u_q'$ are non-negative integers which are referred to as the p-character of the fundamental units. We have $u_i'\leqq u_i$ for $i=1, \cdots, q$.

Next some congruential theorems are obtained which involve the expression $D_z \log \omega(e^v)$; this means the zth derivative at $v=0$. In particular it is proved that there exists in $\Omega(\zeta)$ a fundamental set of units $\delta_1, \cdots, \delta_q$, which satisfy (*) and such that

$$D_{2jp^t} \log \delta_i(e^v)\equiv 0 \pmod{p^{t+1}}\quad (j=1, \cdots, i-1, i+1, \cdots, q)$$

for arbitrary $t\geqq w$, where $w=\max(u_1, \cdots, u_q)$. As an application it is proved that if ε is a unit of $\Omega(\zeta)$ which is congruent to a rational integer (mod $p^{w'+s}$), where $w'=\max(u_1', \cdots, u_q')$, then ε is the p^s-th power of a unit of $\Omega(\zeta)$. *L. Carlitz* (Durham, N.C.).

Šafarevič, I. R. Construction of fields of algebraic numbers with given solvable Galois group. Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 525–578. (Russian)

L'auteur montre que, pour tout corps (de degré fini) k de nombres algébriques et pour tout groupe résoluble G, il existe des extensions K/k, dont le groupe de Galois soit (à l'isomorphie près) G. Il le fait en donnant une méthode inductive de construction de telles extensions, la démonstration de la possibilité de poursuivre cette construction aussi loin que cela est nécessaire se faisant par les méthodes généralisant celles de ses travaux dans le même journal **18** (1954), 261–296, 389–418 [MR **16**, 571, 1001].

Les considérations de la théorie des groupes permettent de simplifier comme suit le problème: si G est une image homomorphe d'un groupe G', la solubilité du problème pour G' entraîne sa solubilité pour G; en combinant ceci avec le théorème d'Ore sur l'existence, dans un groupe résoluble G, d'un sous-groupe invariant nilpotent N et d'un sous-groupe propre G^* tels que $G=G^*/N$, on montre qu'il suffit de savoir résoudre le problème suivant: k/k_0 étant une extension galoisienne de groupe résoluble Γ, scholzienne (mod M), où M est divisible par une puissance suffisemment grande de p, et g étant un p-groupe opéré par Γ, construire une surextension K/k_0 de k/k_0, dont le groupe de Galois soit le produit semi-direct $\Gamma\cdot g$ des Γ et g et qui soit aussi scholzienne (mod M). D'autre part, tout groupe d'exposant divisant p^e opéré par Γ est une image homomorphe d'un groupe de la forme $G_{\Gamma,d,e}^{(c)}$, défini comme suit dans le second travail cité de l'auteur: m étant l'ordre de Γ, considérons le groupe libre engendré par dm générateurs $s_{\xi,i}$ $(\xi \in \Gamma;\ i=1, 2, \cdots, d)$. Posons, en plus, $s_{\xi,i}^{\gamma}=s_{\gamma\xi,i}$. Le groupe opéré par Γ ainsi défini sera noté $G_{\Gamma,d}$, et on notera $G_{\Gamma,d,e}$ le quotient de $G_{\Gamma,d}$ par son sous-groupe caractéristique (donc stable pour Γ) engendré par les puissances p^e-ièmes de ses éléments. $G_{\Gamma,d,e}^{(c)}$ est le quotient de $G_{\Gamma,d,e}$ par le c-ième terme de sa suite centrale descendente, considéré comme opéré par Γ.

k/k_0 étant une extension galoisienne, dont le groupe de Galois soit $\Gamma=G_{k/k_0}$, elle est dite scholzienne (mod M) si: 1) tous les diviseurs premiers de M dans k_0 et tous les idéaux premiers à l'infini de k_0 se décomposent complètement dans k/k_0; 2) tous les facteurs premiers dans k du discriminant de k/k_0 sont de degré 1 dans k/k_0 et leurs normes absolues sont toutes $\equiv 1$ (mod M). Un $\alpha \in k$ est dit

p-invariant dans k/k_0 si, pour tout $\sigma \in \Gamma$, on a $\alpha^\sigma \equiv \alpha$ (mod $k^{(p)}$), où $k^{(p)}$ est le groupe des puissances p-ièmes des éléments du groupe multiplicatif k^* de k, et on appelle classe de p-invariance toute classe (mod $k^*k^{(p)}$), dont les éléments sont p-invariants. G étant une extension (au sens de Schreier) donnée de Γ par un groupe g, une surextension galoisienne K/k_0 de k/k_0 sera dite une G-extension de k/k_0 si G_{K/k_0} peut être identifié avec G de manière que $G_{K/k}$ le soit avec g et que l'homomorphisme canonique de G_{K/k_0} sur G_{k/k_0} le soit avec celui de G sur Γ. Une telle extension est dite une p-surextension si l'ordre de g est une puissance de p. Elle est dite une extension centrale (de k/k_0) si g est contenu dans le centre de G, et elle est dite centrale simple si, en plus, g est un groupe cyclique d'ordre premier. A. Scholz avait montré [Math. Z. **30** (1929), 332–356] que toute p-extension centrale simple de k/k_0 est réalisable si k/k_0 est scholzienne (mod p^h) pour un h suffisemment grand. L'auteur, dans son premier travail cité, a montré que, k/k_0 étant une extension galoisienne scholzienne (mod p^h) et X étant une classe de p-invariance dans k/k_0, la condition nécessaire et suffisante pour que l'on puisse choisir un $\mu \in X$ de manière que $k(\mu^{1/p})/k$ soit scholzienne (mod p^h) est que certains invariants (définis par l'auteur) (χ, X) (où χ parcourt certains caractères de degré 1 et d'ordre p de Γ) et $(X)_h$ soient $=1$. Soient Π un p-groupe de Sylow de Γ, $\bar{k}$ le sous-corps de k appartenant à Π et X une classe de p-invariance dans $k/\bar{k}$. Supposant que k contient les racines p-ièmes de l'unité et que k/k_0 est scholzienne (mod M), où M se divise par une puissance suffisamment grande de p, l'auteur définit, dans le § 1 du présent travail, certains invariants $[\chi, X]$ (comprenant les (χ, X) comme cas particulier), $(X)_M$ (égal à $(X)_h$ si $M=p^h$) et $[X]_\varphi$, et il montre, dans le § 2, que l'égalité à 1 de tous ces invariants est la condition nécessaire et suffisante pour qu'il existe un $\mu \in X$ tel que: 1) $\bar{k}$ soit le plus petit corps intermédiaire de k/k_0 par rapport auquel $K=k(\mu^{1/p})$ soit galoisien; 2) tous les corps conjugués distincts de K/k_0 soient linéairement disjoints (dans leur ensemble) sur k; 3) le composé de tous ces corps (autrement dit le corps de Galois K' de K/k_0) soit, également, scholzien (mod M) sur k_0.

Dans le § 3, l'auteur détermine la condition nécessaire et suffisante pour que les p-extensions centrales arbitraires de k/k_0 puissent se réaliser de manière que (en désignant par K/k_0 l'extension centrale considérée de k/k_0) les conditions précédentes 1), 2) et 3) soient satisfaites. Il est, évidemment, nécessaire pour cela que les invariants $[\chi, X]$, $(X)_M$ et $[X]_\varphi$ soient identiquement égaux à 1 sur tout leur domaine de définition; mais, en plus, certains autres invariants $(X, Y)^{(i)}{}_{\varphi_1,\varphi_2,\cdots,\varphi_{i-1}}$ définis par l'auteur, doivent aussi être $=1$ identiquement, et l'ensemble de ces conditions est suffisant. Les définitions des invariants $[\chi, X]$, $(X)_M$, $[X]_\varphi$, $(X, Y)^{(i)}{}_{\varphi_1,\varphi_2,\cdots,\varphi_{i-1}}$ et les démonstrations des résultats indiqués des §§ 1, 2 et 3 constituent la partie la plus essentielle du travail, mais sont trop compliqués pour qu'on en puisse donner ici même une idée.

Dans le § 4, l'auteur applique le résultat du § 3 pour démontrer l'existence (pour un p fixé) des $\Gamma \cdot G_{\Gamma,d,q}{}^{(c)}$-extensions K/k_0 de k/k_0. Soient $G_{\Gamma,d,q}{}^{(c,r)}$ le quotient de $G_{\Gamma,d,q}{}^{(c)}$ par le groupe des éléments σ de son centre tels que $\sigma^{p^{q-r}}=1$, et Z_r le noyau de l'homomorphisme naturel de $G_{\Gamma,d,q}{}^{(c,r)}$ sur $G_{\Gamma,d,q}{}^{(c,r-1)}$. Z_r est un groupe abélien opéré par Γ. Π étant un p-groupe de Sylow de Γ, soit $Z_{r,0}=Z_r$, $Z_{r,1}=[Z_{r,0}, \Pi]$, $\cdots$, $Z_{r,i}=[Z_{r,i-1}, \Pi]$, $\cdots$ la suite des commutateurs successifs de Z_r avec Π, et soit $G_{\Gamma,d,q}{}^{(c,r,i)}=G_{\Gamma,d,q}{}^{(c,r)}/Z_{r,i}$. $Z_r/Z_{r,i}$ est un groupe abélien opéré par Π, et on notera $\mathfrak{A}$ ce groupe avec Π comme groupe d'opérateurs (même si d'autres éléments de Γ préservent $Z_{r,i}$ et opérent, ainsi, dans $Z_r/Z_{r,i}$). L'auteur définit, à l'aide de $G_{\Gamma,d,q}{}^{(c,r,i)}$ en tant qu'extension de $\mathfrak{A}$, un certain groupe $G_{\Gamma,d,q}{}^{(c,r,i)*}$ opéré par Γ tel que, en vertu d'un théorème de Gaschütz [J. Reine Angew. Math. **190** (1952), 93–107; MR **14**, 445], $\Gamma \cdot G_{\Gamma,d,q}{}^{(c,r,i)*}$ possède, comme image homomorphe, un groupe de la forme $\Gamma \cdot G_{\Gamma,d,q}{}^{(c,r,i)}$. D'autre part, il existe un homomorphisme naturel de $G_{\Gamma,d,q}{}^{(c,r,i)*}$ sur $G_{\Gamma,d,q}{}^{(c,r,i-1)*}$, et le noyau N^* de cet homomorphisme est un groupe, contenu dans le centre de $G_{\Gamma,d,q}{}^{(c,r,i)*}$ (donc abélien) et engendré, en tant que sous-groupe stable de ce groupe, par un de ses sous-groupes abstraits N, où tout opérateur appartenant à Π induit l'identité .

L'auteur prouve, en employant certains homomorphismes de $G_{\Gamma,d,q}{}^{(c,r,i)*}$ dans $G_{\Gamma,\delta,q}{}^{(c,r,i)*}$ $(d>\delta)$ analogues à ceux de son second travail cité, que si d est suffisemment grand par rapport à δ et si Q/k_0 est une $\Gamma \cdot G_{\Gamma,d,q}{}^{(c,r,i)*}$-extension de k/k_0, qui est scholzienne (mod M), il existe une sous-extension K/k_0 de Q/k_0, qui est une $\Gamma \cdot G_{\Gamma,\delta,q}{}^{(c,r,i)*}$-extension de k/k_0 telle que tous les invariants $[\chi, X]$, $(X)_M$, $[X]_\varphi$ et $(X, Y)_{\varphi_1,\varphi_2,\cdots,\varphi_{i-1}}{}^{(i)}$ de K/k_0 soient égaux à 1 chaque fois qu'ils ont un sens. De lors, la possibilité de construire une $\Gamma \cdot G_{\Gamma,d,q}{}^{(c)}$-extension de k/k_0 où k/k_0 est scholzienne modulo un entier M_c divisible par une puissance suffisemment grande (et dépendant de c) de p, se démontre par l'induction suivante: on suppose, pour un $x<c$ pour un $r<q$ et pour un i, que, pour tout d, il existe des $\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r,i)*}$-extensions scholziennes (mod M_c) de k/k_0. De lors, il en existe de telles extensions $\bar{K}/k$, où, en plus, tous les invariants indiqués sont $=1$ identiquement. Mais alors, si $\bar{k}$ est le corps intermédiaire de k/k_0 appartenant à Π, on peut, en vertu du § 3, construire, pour tout d, une extension centrale $Q/\bar{k}$ de $\bar{K}/\bar{k}$ telle que, si K est le corps de Galois de Q/k_0, K/k_0 soit une $\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r,i+1)*}$-extension scholzienne (mod M_c) de k/k_0. On a $G_{\Gamma,d,q}{}^{(x,r,0)*}=G_{\Gamma,d,q}{}^{(x,r)}$ et, pour quelque m, $G_{\Gamma,d,q}{}^{(x,r,m)*}=G_{\Gamma,d,q}{}^{(x,r+1)*}$. Ainsi, comme $\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r,m)*}$ $=\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r+1)*}$ admet $\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r,m)}=\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r+1)}$ comme image homomorphe, l'existence, pour tout $x<d$, d'une $\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r)}$-extension scholzienne (mod M_c) de k/k_0 entraîne celle d'une $\Gamma \cdot G_{\Gamma,d,q}{}^{(x,r+1)}$-extension de k/k_0 de même forme. Comme $G_{\Gamma,d,q}{}^{(x,0)}=G_{\Gamma,d,q}{}^{(x)}$ et $G_{\Gamma,d,q}{}^{(x,q)}=$ $G_{\Gamma,d,q}{}^{(x+1)}$, l'existence, pour tout d, d'une $\Gamma \cdot G_{\Gamma,d,q}{}^{(x)}$ -extension scholzienne (mod M_c) de k/k_0 entraîne celle d'une $\Gamma \cdot G_{\Gamma,d,q}{}^{(x+1)}$-extension de k/k_0 de même forme, d'où, en vertu de $G_{\Gamma,d,q}{}^{(0,0,0)}=1$, résulte l'existence des $\Gamma \cdot G_{\Gamma,d,q}{}^{(c)}$-extensions de k/k_0 scholziennes (mod M_c).

Le travail se termine par un complément, où est démontrée une généralisation au cas non-galoisien du résultat du travail dans Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 327–334 [MR **16**, 572] de l'auteur, dont il a besoin dans une démonstration du § 2. *M. Krasner.*

Yamazaki, Keijiro. On fibre spaces in the algebraic number theory. J. Math. Soc. Japan **7** (1955), 182–201.

Let k be a finite algebraic number field, K the field obtained by adjoining all roots of unity to k and S the set of all non-archimedian prime divisors of K. For any $x \neq 0$ in k, the author defines a function $x(P)$ on S as follows (ν_P=an exp. valuation for P): $x(P)=0$ if $\nu_P(x)>0$, $x(P)=\infty$ if $\nu_P(x)<0$ and $x(P)=\zeta$ if $\nu_P(x)=0$ and if ζ denotes the unique root of unity with order prime to p $(P|p)$ such that $\nu_P(x-\zeta)>0$. Let $\bar{S}$ be the space obtained by identifying those P, P' in S for which $x(P)=x(P')$ for

all $x\neq 0$ in k and let $\bar{x}(\bar{P})$ denote the function on $\bar{S}$ obtained from $x(P)$ in an obvious way. $\bar{S}$ is made a topological space with respect to its weakest T_1-topology. The author considers the space $\bar{S}$ and the functions $\bar{x}(\bar{P})$ as analogues of an algebraic curve and rational functions on it. To obtain what correspond to algebraic varieties in general, the author then introduces, by abstracting the essential properties of $\bar{S}$ and $\bar{x}(\bar{P})$ above, the notion of a W-variety (V, F) which consists of a suitable T_1-space V and a family of functions defined on open subsets of V, taking values in the set $\overline{W}$ consisting of 0, ∞ and all roots of unity and having similar properties to those of $\bar{x}(\bar{P})$. Rational mappings of such W-varieties and fibre W-spaces are also defined, the latter being fibre spaces whose base spaces, fibres, structural groups are all W-varieties, with rational mappings among them.

For a special type of W-variety V, it is then proved, following the analogy in algebraic geometry, that the multiplicative group of all classes of (rational) principal fibre W-spaces, having the base space V and the group W, is naturally isomorphic with the (rational) divisor-class group of V which is defined for V similarly as the divisor-class group of an algebraic curve. Applying this to $V=\bar{S}$, it then follows that the group attached to the space $\bar{S}$ is naturally isomorphic with the ideal-class group of k, and a similar result can be also obtained for "Strahl" ideal-class groups of k. *K. Iwasawa.*

See also: Shah, p. 141.

Geometry of Numbers, Diophantine Approximation

Lekkerkerker, C. G. On the determinant of an asymmetric hyperbolic region. Ann. Mat. Pura Appl. (4) 38 (1955), 253–266.

Let $K_{a,b}$ denote the plane domain defined by $-a\leq xy\leq b$, $a>0$, $b>0$; and let $\Delta(a, b)$ denote its critical determinant with respect to lattices having a fixed point at the origin. Observing that there is no loss of generality in taking $a=1$, $b\geq 1$, the author proves that

$$\Delta(1, b)\geq \min\{\sqrt{(\beta^2+4b)},\ \sigma\sqrt{(b^2+4b\sigma)}\},$$

where $\beta=-[-b]$, $\sigma=b/[b]$; the equality sign holding if and only if, either $\sigma\sqrt{(b^2+4b\sigma)}\leq\sqrt{(\beta^2+4b)}$, or $(\beta+2)/(\beta+1-b)$ is an integer. This result is a sharper form of a theorem of B. Segre [Duke Math. J. 12 (1945), 337–365; MR 6, 258]. Defining b_0 by the relations $b_0^4=4(1+b_0)/5$, $b_0>1$, it is shown that the (fully automorphic) star body $K_{1,b}$ is boundedly irreducible for $1<b<b_0$. *J. H. H. Chalk* (London).

Val'fiš, A. Z. On lattice points in multidimensional ellipsoids. XVI. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 20 (1954), 1–20. (Russian)

Let $P_{2k}(x)=A_{2k}(x)-V_{2k}(x)$, where $A_{2k}(x)$ is the number of lattice points in the sphere $z_1^2+\cdots+z_k^2=x$, and $V_{2k}(x)=\pi^k x^k/\Gamma(k+1)$ is its volume. In previous papers of this series [same Trudy 15 (1947), 275–296, 297–322; 16 (1948), 215–230; 17 (1949), 245–258; MR 13, 919] the author derived properties of the functions P_{2k}, ϱ_{2k}, defined by

$$P_{2k}=\limsup_{n\to\infty}\frac{P_{2k}(n)}{M_{2k}\,n^{k-1}},\quad \varrho_{2k}=\liminf_{n\to\infty}\frac{P_{2k}(n)}{M_{2k}\,n^{k-1}},$$

where $M_{2k}=\pi^k/2\Gamma(k)$. In the present paper he derives analogous results for the functions $P_{2k,a}$, $\varrho_{2k,a}$ defined by

$$P_{2k,a}=\limsup_{\substack{n\to\infty\\ n\equiv a\ (\mathrm{mod}\ 2)}}\frac{P_{2k}(n)}{M_{2k}n^{k-1}},$$

$$\varrho_{2k,a}=\liminf_{\substack{n\to\infty\\ n\equiv a\ (\mathrm{mod}\ 2)}}\frac{P_{2k}(n)}{M_{2k}\,n^{k-1}}.$$

In particular, he shows that (a) $P_{2k,a}+\varrho_{2k,1-a}=2$, (b) $P_{2k,0}=P_{2k}$, (c) $\varrho_{2k,1}=2-P_{2k}=\varrho_{2k}$. *W. H. Simons.*

Pipping, Nils. Halbregelmässige Kettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen. Acta Acad. Abo. 20 (1955), no. 1, 9 pp.

For a positive integer D, not a perfect square, the author gives a procedure for determining from the regular continued fraction for $\sqrt{D}$ a particular semi-regular continued fraction for $\sqrt{D}$. Some properties of this semi-regular continued fraction are noted, and for $1<D\leq 1000$, the 101 such semi-regular continued fractions which are not diagonal continued fractions are tabulated. *W. T. Scott* (Evanston, Ill.).

Szüsz, P. Bemerkungen zur Approximation einer reellen Zahl durch Brüche. Acta Math. Acad. Sci. Hungar. 6 (1955), 203–212. (Russian summary)

For irrational α and $0\leq\delta\leq 1$, let $M(\alpha, \delta, n)$ be the number of positive integers $q\leq n$ for which $|q\alpha-p|<q^{\delta-1}$ for suitable integer p. A proof due to Turán is given, showing that $M(\alpha, \delta, n)>n^\delta$ for $\delta>0$. It is also shown that for $\alpha=(1+5^{\frac12})/2$ there are infinitely many n for which $M(\alpha, \delta, n)<K(\delta)n^\delta$ for some constant K. Moreover, for $\varepsilon>0$ there is an $N_0=N_0(\varepsilon, \delta)$ such that for every real α there is an n between N and $N^{1+\varepsilon}$ for which $|n\alpha-m|<n^{-\delta}$ for suitable m, if $N>N_0$, but that there is no such n between N and $g(N)N^{1+\delta}$, for infinitely many N, if $g(x)\downarrow 0$ as $x\to\infty$. *W. J. LeVeque* (Ann Arbor, Mich.).

See also: Popken, p. 117.

ANALYSIS

Černikov, S. N. Positive and negative solutions of systems of linear inequalities. Dokl. Akad. Nauk SSSR (N.S.) 99 (1954), 913–916. (Russian)

Conditions using signs of minors for the existence of solutions, with prescribed sign of certain unknowns, of a system of linear inequalities or equations [see Černikov, Uspehi Mat. Nauk (N.S.) 8 (1953), no. 2(54), 7–73; MR 15, 293]. *T. S. Motzkin* (Los Angeles, Calif.).

Collatz, L. Über monotone Systeme linearer Ungleichungen. J. Reine Angew. Math. 194 (1955), 193–194.

If in a real matrix A, $a_{jk}\leq 0$ for $j\neq k$ and $Ax_0>0$ for some $x_0>0$ then $Ax\geq 0$ implies $x\geq 0$. Simple proof of this and a related theorem of Ostrowski [Comment. Math. Helv. 10 (1937), 69–96]. *T. S. Motzkin* (Los Angeles, Calif.).

Rubinštein, G. Š. The problem of the boundary point of the intersection of an axis with a polyhedron and its application to the investigation of a finite system of linear inequalities. Dokl. Akad. Nauk SSSR (N.S.) 100 (1955), 627–630. (Russian)

A general finite linear-inequalities or optimization problem is considered as the problem of finding a hyper-

plane passing above a given convex polyhedron and meeting a given straight line as low as possible. Fourier's descent method [cf. Zuhovickiĭ, same Dokl. (N.S.) **79** (1951), 561–564; MR **13**, 285; and Šteinberg, Dopovidi Akad. Nauk Ukrain. RSR **1952**, 167–173; MR **16**, 558] is described in some detail in this dual interpretation.
T. S. Motzkin (Los Angeles, Calif.).

Watson, G. N. Schur's inequality. Math. Gaz. **39** (1955), 207–208.

The inequality is
$$x^\mu(x-y)(x-z)+y^\mu(y-z)(y-x)+z^\mu(z-x)(z-y)\geqq 0$$
whenever x, y, z are positive and μ is real. The author discussed it in an earlier note [Math. Gaz. **37** (1953), 244–246; MR **15**, 407] pointing out the desirability of a symmetrical proof, which he could give only for $-1\leqq\mu\leqq 0$, and $\mu=1$, 2 (hence for $\mu=-2$, -3). He now gives a symmetrical proof for $0\leqq\mu\leqq 1$ (and hence for $-2\leqq\mu\leqq -1$).
R. P. Boas, Jr. (Evanston, Ill.).

Salzmann, Helmut, und Zeller, Karl. Singularitäten unendlich oft differenzierbarer Funktionen. Math. Z. **62** (1955), 354–367.

The authors are concerned with two kinds of singularity that an infinitely differentiable function can have: (A) points about which its Taylor series diverges and (B) points about which the series converges to the wrong values. They discuss various methods for constructing functions with prescribed singularities and give simple examples of functions which have singularity (A) at every point or singularity (B) at every point. They give a simplified proof of Zahorski's characterization of sets which can consist of singularities of the two kinds [Fund. Math. **34** (1947), 183–245; MR **10**, 23]. They discuss some related results and give a historical account of the subject and an extensive bibliography. *R. P. Boas, Jr.*

Moisil, Gr. C. Polynômes associés aux expressions différentielles bilinéaires à coefficients constants. Com. Acad. R. P. Române **2** (1952), 129–131. (Romanian. Russian and French summaries)

The author considers bilinear differential expressions E with constant coefficients, i.e. sums of terms of the form
$$a\frac{\partial^r\varphi}{\partial x_{i_1}\cdots\partial x_{i_r}}\cdot\frac{\partial^s\psi}{\partial x_{j_1}\cdots\partial x_{j_s}}.$$
Setting $\varphi^*=\varphi(x_1^*, \cdots, x_n^*)$ and $\psi^{**}=\psi(x_1^{**}, \cdots, x_n^{**})$ in the previous form and associating with
$$a\frac{\partial^r\varphi^*}{\partial x_{i_1}^*\cdots\partial x_{i_r}^*}\cdot\frac{\partial^s\psi^{**}}{\partial x_{j_1}^{**}\cdots\partial x_{j_s}^{**}}$$
the monomial $aX_{i_1}^*X_{i_2}^*\cdots X_{i_r}^*X_{j_1}^*\cdots X_{j_s}^*$, a polynomial $P(X_1^*, \cdots, X_n^*, X_1^{**}, \cdots, X_n^{**})$ is associated with the previous bilinear form E. A one-to-one correspondence is exhibited between the polynomials P and the differential expressions E, in which x_i^* and x_i^{**} have been identified with x_i. Properties of bilinear differential expressions are obtainable from the corresponding properties of the polynomials P. *E. Grosswald.*

Theory of Sets, Functions of Real Variables

Have

★ **Bachmann, Heinz. Transfinite Zahlen.** Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 1. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. vii+204 pp. DM 29.80.

This is a report on the present state of the results and problems in the theory of transfinite numbers (ordinal numbers and powers, but not general order types). It is based on the Zermelo-Fraenkel axiom system, but is written in the language of naive set theory. Arithmetical questions are treated in detail, whereas axiomatic questions are considered less extensively. Proofs are given for most of the theorems cited, and when the axiom of choice is used, this is expressly indicated.

In the first chapter, after some introductory remarks concerning naive set theory, the mathematical foundations problem, and the various schools of mathematics, the usual axioms of set theory are presented and the fundamental notions of equivalence, similarity, well-ordering, transfinite induction, and transfinite numbers are introduced.

The next two chapters are devoted to the theory of ordinal numbers, developed for the most part without the use of the axiom of choice. Ordinal-valued functions of ordinal numbers and the arithmetic of ordinal numbers are the main topics, and there are, besides a wealth of more standard material, recent results concerning normal functions, regressive functions, sequences of continuous functions, rarified classes of ordinal numbers, and permutations of sequences of ordinal numbers. A simple treatment of the theory of principal ordinal numbers is also given.

The fourth chapter deals with the arithmetic of powers without the axiom of choice, and is followed by a chapter which develops further the arithmetic of cardinal numbers with the aid of this axiom. A considerable amount of what is known concerning sums, products, powers, and inequalities involving alephs is to be found here, as well as a number of theorems equivalent to the axiom of choice or to the generalized continuum hypothesis.

The sixth chapter is devoted to the continuum as an order type (with regard to its well-ordering and its power) and to the second number class and the subject of distinguished sequences. A few connections between the continuum and the theory of point sets are given, but the book is not concerned with a systematic presentation of the applications of transfinite numbers to other fields. The question of the formal representation of ordinal numbers is discussed briefly, and there is a section on alternatives to the axiom of choice. The last chapter is devoted to inaccessible ordinal and cardinal numbers. The work is provided with a bibliography and an index.
F. Bagemihl (Notre Dame, Ind.).

von Krbek, Franz. Wohlordnung. Acta Math. **93** (1955), 313–316.

The author claims that the axiom of choice in the special form where it applies to one set is independent of the remaining axioms of Zermelo; this, however, is not so [cf. Fraenkel, Enseignement Math. **34** (1935), 32–51].
F. Bagemihl (Notre Dame, Ind.).

Seki, Setsuya. On transfinite inferences. Comment. Math. Univ. St. Paul. **4** (1955), 43–45.

The author supplies a proof of the equivalence, under Gödel's axioms (without the axiom of choice), of the following propositions about sets: well-ordering theorem (W), axiom of choice (A), Zorn's lemma (Z), comparability theorem (C). The implications $(W)\to(A)$ and $(Z)\to(C)$ are obvious. The implications $(A)\to(Z)$ and $(C)\to(W)$ are established by indirect proofs, the contradiction obtained in each case being that a certain set must be a proper class. Some misprints: footnote 3) should refer

to (C3), not (C1); the first A on the last line of p. 44 should be 2^A, and a corresponding change should be made on p. 45, line 8. *L. Gillman* (Lafayette, Ind.).

Gillman, Leonard. Remarque sur les ensembles η_α. C. R. Acad. Sci. Paris **241** (1955), 12–13.

Any two $\eta_{\alpha+1}$-sets of power $\aleph_{\alpha+1}$ are similar; and there exists an $\eta_{\alpha+1}$-set of power $\aleph_{\alpha+1}$ if, and only if, $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ [Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, pp. 181–182]. The author shows that, if $2^{\aleph_\alpha}\neq\aleph_{\alpha+1}$, then there exist two dissimilar $\eta_{\alpha+1}$-sets of power $2^{\aleph_\alpha}$. If $\delta\geqq\beta+2$ and $\aleph_\delta\geqq 2^{\aleph_\beta}$, then there exist two dissimilar $\eta_{\beta+1}$-sets of power $\aleph_\delta$. *F. Bagemihl* (Notre Dame, Ind.).

Kurepa, Đuro. On reflexive symmetric relations and graphs. Slovenska Akad. Znan. Umet. Razred Mat. Fiz. Tehn. Vede. Ser. A. **4** (1953), no. 4, 65–92. (Serbo-Croatian and English)

Notation: kG is the power of G, $k_cG=\sup_X kX$ and $k_sG=\sup_Y kY$, X and Y running over all connected and disconnected subgraphs respectively of G. The principal result is that, for any graph G, $kG\leqq(2k_sG)^{k_cG}$. Furthermore, for any $\aleph_\alpha$ and any cardinal number $n\leqq 2^{\aleph_\alpha}$ there is a graph $g(\aleph_\alpha, n)$ such that $k_cg\leqq\aleph_\alpha$, $k_sg\leqq\aleph_\alpha$, and $kg=n$; for $n=2^{\aleph_\alpha}$ there is a graph M_α such that $k_cM_\alpha=\aleph_\alpha=k_sM_\alpha$ and $kM_\alpha=2^{\aleph_\alpha}$. As a consequence of this result and the continuum hypothesis, the author answers a question of Sierpiński, namely that there does not exist a graph G of power $\geqq\aleph_2$ such that each infinite, non-denumerable subgraph of G contains two connected points and two disconnected points. This question however has been answered already by Erdös [Univ. Nac. Tucumán. Rev. Ser. A. 3 (1942), 363–367; MR **5**, 151]. A paper by Dushnik and Miller [Amer. J. Math. **63** (1941), 600–610; MR **3**, 73] should be added to the references. In particular, the partially ordered set $(M_\alpha;\leqq)$ given on page 91 of the paper under review, is a generalization of an example in the paper of Dushnik and Miller. *S. Ginsburg.*

Mateos Gonzalez, Isaac. On a definition of finite sets. Euclides, Madrid **15** (1955), 150. (Spanish)

Horn, Alfred. A characterization of unions of linearly independent sets. J. London Math. Soc. **30** (1955), 494–496.

Let k and X be respectively any cardinal and any vector space over a division ring; a subset S of X is said to have the R_k-property provided $|T|\leqq k\times\text{rank } T$ for every $T\subseteq S$; rank T denotes the supremum of cardinals of linearly independent subsets of T. Theorem (conjectured by K. F. Roth and R. Rado; the converse of the theorem is obvious): If k is a positive integer and if a set S of vectors has the R_k-property, then S is the union of k disjoint linearly independent sets. For infinite k, the statement does not hold. E.g. let X be a vector space over the rationals; then X has the $R_{\aleph_0}$-property; the converse holds only if $|X|\leqq\aleph_1$ [cf. Erdös and Kakutani, Bull. Amer. Math. Soc. **49** (1943), 457–461; MR **4**, 249]. *G. Kurepa* (Zagreb).

Schmidt, Jürgen. Lexikographische Operationen. Z. Math. Logic Grundlagen Math. **1** (1955), 127–171.

This paper discusses lexicographic ordering in the full cartesian product $E=\mathsf{P}_{i\in I} E_i$ of partially ordered sets. The first half of the paper surveys the many definitions already given for "lexicographic order" in the set E, compares the relative strengths of these orderings, and discusses their various defects; of these the most noteworthy is that the product need not to be totally (or well) ordered even if all the factors and the index set are.

The last half of the paper is an axiomatic discussion of the properties of a lexicographic order which cause this unfortunate result. We quote some of these results in the simple special case where every E_i contains a totally ordered two-element subset. If L is a partial ordering of E which gives the natural lexicographic order to each pair of functions f, g which differ only on a finite subset of E (i.e., if $f\leqq g$ means $f(i)\neq g(i)$, then there is $j\leqq i$ such that $f(j)<g(j)$), then L well orders E if and only if all E_i are well ordered and I is well ordered in both directions; hence finite. Under the above hypothesis on L, there is such a total order L in E if and only if I and all E_i are totally ordered; some extra isomorphism conditions then force I to be well ordered. *M. M. Day* (Urbana, Ill.).

Četković, Simon. Les nombres transcendants et la différentiabilité d'une famille de fonctions. Formation d'un ensemble des nombres transcendants. Bull. Soc. Math. Phys. Serbie **6** (1954), 93–101. (Serbo-Croatian. French summary)

Let S be a finite set of transcendental numbers. The author constructs a real-valued function on the entire line that is discontinuous at all rational points, is differentiable at every point of S, and is also differentiable on a certain dense set. There is also a dense set of transcendental points where the function is continuous but not differentiable. *E. Hewitt* (Princeton, N.J.).

Bonati Savorgnan, Carlo. Sulla differenziabilità secondo Stolz delle funzioni composte. Ann. Univ. Ferrara. Sez. VII. (N.S.) **3** (1954), 17–24.

Using the method of his previous paper [Rend. Sem. Mat. Univ. Padova **22** (1953), 258–264; MR **15**, 410] and a result of J. Cecconi [Richerche Mat. **1** (1952), 317–324; MR **14**, 737], the author establishes sufficient conditions for the differentiability (in the ordinary sense or in the Stolz sense, respectively) of composite functions (a) $F(t)=f(x(t), y(t), z(t))$ and (b) $F(t,\tau)=f(x(t,\tau), y(t,\tau), z(t,\tau))$. In both cases $f(x, y, z)$ is assumed in a three-dimensional interval to satisfy a uniform Lipschitz condition and to have first derivatives which are continuous with respect to the pairs of variables. In particular, under these assumptions he proves the theorem: The functions $x(t,\tau)$, $y(t,\tau)$, $z(t,\tau)$, defined in a rectangle S: $\alpha\leqq t\leqq\beta$, $\gamma\leqq t\leqq\delta$, are supposed to be almost everywhere in S differentiable in the Stolz sense and to be absolutely continuous with respect to t for almost every τ of $[\gamma,\delta]$. Moreover, it is assumed that, almost everywhere in S,

$$x_t^2(t,\tau)+y_t^2(t,\tau)+z_t^2(t,\tau)>0,$$

and that the point $[x(t,\tau), y(t,\tau), z(t,\tau)]$ is in the interior of R for almost every (t,τ) of S. Then the composite function (b) is differentiable in the Stolz sense almost everywhere in S. *A. Rosenthal* (Lafayette, Ind.).

Ostrowski, Alexandre. Note sur les dérivées uniformes et les différentielles totales. Comment. Math. Helv. **29** (1955), 298–300.

F. Severi [Ann. Mat. Pura Appl. (4) **13** (1934), 1–35] proved, under certain restrictions on $f(x, y)$, the existence of its total derivative (au sens de Stolz). The author [Comment. Math. Helv. **15** (1943), 222–226; MR **5**, 233] showed the existence of this derivative under somewhat different conditions. In the present paper remarks are made on the interrelations of these two sets of sufficient conditions. *F. G. Dressel* (Durham, N.C.).

Foiaş, Ciprian. Une nouvelle démonstration d'un théorème de Denjoy. Com. Acad. R. P. Romîne 4 (1954), 187–190. (Romanian. Russian and French summaries)

The author gives a new proof of the following theorem of Denjoy: Let $f(x)$ be defined over the interval Δ. If for every $x \in \Delta$, $f(x)$ has a right and a left derivative, then $f(x)$ is differentiable on Δ, except possibly on a denumerable set. *E. Grosswald* (Philadelphia, Pa.).

Czipszer, J., and Gehér, L. Extension of functions satisfying a Lipschitz condition. Acta Math. Acad. Sci. Hungar. 6 (1955), 213–220. (Russian summary)

Let S be a subset of a metric space X and f a real-valued function on S. Two extension theorems are proved. (I) If f satisfies a Lipschitz condition on S with constant K, then the function F defined by $F(x) = \inf\{f(y) + K \operatorname{dist}(x, y) : y \in S\}$ is an extension of f and satisfies a Lipschitz condition on X with constant K. [The authors point out in a note added in proof that their proof is the same as one given by S. Banach, Introduction to the theory of real functions (Polish), Warszawa-Wrocław, 1951, pp. 121–122; MR 13, 216.] The second theorem requires that S be closed, but only assumes that for each $x \in S$ there are constants $K_x \geq 0$ and $\delta_x > 0$ such that if $\operatorname{dist}(x, y) < \delta_x$ then $|f(x) - f(y)| \leq K_x \operatorname{dist}(x, y)$. It is remarked that both results remain valid if $\operatorname{dist}(x, y)$ is replaced by $\omega(x, y)$ where ω is a continuous monotone increasing subadditive function with $\omega(0) = 0$. *R. G. Bartle* (Urbana, Ill.).

Marczewski, E. Remarks on sets of measure zero and the derivability of monotonic functions. Prace Mat. 1 (1955), 141–144. (Polish. Russian and English summaries)

Construction of a purely discontinuous monotonic function $f(x)$ which does not possess a finite derivative at any point of an arbitrarily prescribed set of measure zero. *F. A. Behrend* (Melbourne).

Zaidman, S. Sur une classe de congruences. Com. Acad. R. P. Romîne 4 (1954), 463–469. (Romanian. Russian and French summaries)

Let (K) be a congruence (one-parameter family?) of plane curves, given by $y = F(x, t)$, where the parameter t stands for the ordinate at $x = 0$. If $y_1 = F(x, t_1)$, $y_2 = F(x, t_2)$ satisfy $y_1 - y_2 = t_1 - t_2$ for all x, then y_1 and y_2 are said to be parallel. The set (K) is divided into equivalence classes of parallel curves. The following are some characteristic results: $F(x, t)$ is continuous with respect to t and, for fixed x, F is almost everywhere differentiable with respect to t; the set of equivalence classes of parallel curves is not denumerable, unless it reduces to a single class. It seems to the reviewer that the assumptions stated are incomplete and he was not able to follow the author's argument. *E. Grosswald* (Philadelphia, Pa.).

Poenaru, Valentin. Propriétés métriques des congruences. Com. Acad. R. P. Romîne 5 (1955), 295–301. (Romanian. Russian and French summaries)

Using results of the paper reviewed above, the author studies some metric properties of such congruences. Let E be a measurable plane set, let $y = f(x, t)$, where the parameter t is the ordinate y_0 at $x = 0$, and let $E(t)$ be the projection on the x-axis of the intersection of $y = f(x, t)$ with E. The measure of $E(t)$ is then studied as a function of t. The main result of the paper seems to be the following statement: Let $y = f(x) + t$, where $f(x)$ is a continuous function. Let $F(x, y)$ be summable and vanish outside the domain D bound by $x = a$, $x = b$, $y = f(x) + t_0$, $y = f(x) + T$. Then $F(x, y(x) + t)$ is summable for almost all t and $\iint_D F\,dx\,dy = \int_{t_0}^{T} dt \int_a^b F(x, y(x) + t)\,dx$. The reviewer does not see why this result should not follow directly from Fubini's theorem, nor why F is required to vanish outside D. *E. Grosswald* (Philadelphia, Pa.).

Cinquini, Silvio. Sopra il piano tangente a una superficie. Boll. Un. Mat. Ital. (3) 10 (1955), 400–412.

Theory of Measure and Integration

Poenaru, Valentin. Une nouvelle démonstration du théorème de recouvrement de Vitali. Com. Acad. R. P. Romîne 4 (1954), 191–194. (Romanian. Russian and French summaries)

In the Vitali theorem demonstrated in this note, the basic set M is covered by a family of closed sets F such that, for each $\varepsilon > 0$, each element P of M belongs to a set F with $|F| < \varepsilon$ ($|F|$ = measure of F) and to every F there exists one of a set of parallelepipeds A such that $|F|/|A(F)| > \theta$, θ independent of F, the parallelepipeds A being regular in the sense that the ratio of the lengths of any two edges is between k and $1/k$, k fixed. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Vikraman, V. An approach to the Radon-Nikodým theorem. Math. Student 23 (1955), 1–26.

A discussion of some well-known results concerning the lattice-theoretic structure of the set of all bounded signed measures on a Boolean σ-algebra of sets. *P. R. Halmos.*

Tsurumi, Shigeru. On general ergodic theorems. Tôhoku Math. J. (2) 6 (1954), 264–273.

Let (X, β, μ) be a σ-finite measure space and let $\{f_j(x)\}$ and $\{g_j(x)\}$ be two sequences of measurable functions on X. Then $\{f_j(x)\}$ and $\{g_j(x)\}$ are said to have the combined homogeneity property (c.h.p.) if, for any Borel set A in the $2m$-dimensional Euclidean space E_{2m},

$$\mu(\{x \mid (f_{n_1+h}(x), g_{n_1+h}(x), \cdots, f_{n_m+h}(x), g_{n_m+h}(x)) \in A\})$$

is independent of h. The sequences $\{f_j(x)\}$ and $\{g_j(x)\}$ are said to have the weak combined homogeneity property (w.c.h.p.) if there exists a K such that for any Borel set A in E_{2m} we have

$$\begin{aligned}\limsup_n \frac{1}{n} \sum_{h=0}^{n-1} \mu(\{x \mid (f_{n_1+hk}(x),\ & g_{n_1+hk}(x), \cdots, f_{n_m+hk}(x), g_{n_m+hk}(x)) \in A\}) \\ &\leq K \cdot \mu(\{x \mid (f_{n_1}(x), g_{n_1}(x), \cdots, f_{n_m}(x), g_{n_m}(x)) \in A\}).\end{aligned}$$

The following two theorems are proved. Theorem 1. Let $\{f_j(x)\}$ and $\{g_j(x)\}$ have the c.h.p. Let $f_1(x)$ be μ-integrable and $g_1(x) > 0$ and let $\sum_{j=1}^{\infty} \min(g_j(x), k) = \infty$ for some positive k. Let also $p(j)$ and $q(j)$ be two periodic functions of integers such that $q(j) > 0$ for $j = 1, 2, \cdots$. Then

$$\lim \sum_{j=1}^{n} f_j(x) p(j) \Big/ \sum_{j=1}^{n} g_j(x) q(j)$$

exists and is finite p.p. Theorem 2. Let μ be finite and let $f_j(x)$ be μ-integrable and $g_j(x) > 0$ for $j = 1, 2, \cdots$. Let $p(j)$ and $q(j)$ be as in Theorem 1. Then, if $\{f_j(x)\}$ and $\{g_j(x)\}$ have the w.c.h.p.,

$$\lim \sum_{j=1}^{n} f_j(x) p(j) \Big/ \sum_{j=1}^{n} g_j(x) p(j)$$

exists and is finite p.p. Theorem 1 includes as a special case Doob's ergodic theorem [Duke Math. J. 6 (1940), 290–306; MR 1, 344], Hopf's ergodic theorem [Ergodentheorie, Springer, Berlin, 1937, p. 49], and a theorem of Khintchine [Mat. Sb. 41 (1934), 14–15]. Theorem 2 includes as a special case a part of a theorem of Ryll-Nardzewski [Studia Math. 12 (1951), 65–73; MR 13, 757]. The method used in the proofs of theorems 1 and 2 is inspired by those of Doob in the paper cited above.

Y. N. Dowker (London).

Sobolev, V. I. On a partially ordered measure of sets, measurable functions, and certain abstract integrals. Trudy Voronežsk. Gos. Univ. 33 (1954), 21–35. (Russian)

Let X be a partially ordered ring and $\{e_\lambda\}$, $-\infty<\lambda<+\infty$, some resolution of the identity. This generates a completely additive internal function $e(\Delta)$ with values in X defined by:

$e(\Delta)=e_{\alpha+0}-e_\alpha$ if Δ consists of a single point α,
$e(\Delta)=e_\mu-e_{\lambda+0}$ if $\Delta=(\lambda, \mu)$.

This function, as in the construction of linear Lebesgue measure, may be extended to open and closed sets, allowing one to introduce in the usual way the concepts of outer and inner measure of an arbitrary linear set and to establish the concepts of measurability and measure.

The constructed partially ordered measure of linear sets generates in the usual way the notion of a partially ordered Lebesgue-Stieltjes integral $\int_E f(\lambda)de_\lambda$, where $f(\lambda)$ is a real measurable (relative to the introduced measure) function on a linear set measurable in the considered sense.

The results of the author contain as a special case the theory of operator measures and operator integrals of A. I. Plesner [Plesner and Rohlin, Uspehi Mat. Nauk (N.S.) 1 (1946), no. 1(11), 71–191; MR 9, 43].

P. I. Romanovskiĭ (RŽMat 1955, no. 5162).

Ul'yanov, P. L. Some questions of *A*-integration. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 1077–1080. (Russian)

A measurable real-valued function f on $[a, b]$ is said to be A-integrable if

$$m\{x: x \in [a, b], |f(x)|<n\}=o(n^{-1}) \tag{1}$$

and

$$\lim_{n\to\infty}\int_a^b \min[\max(f(x), -n), n]dx=(A)\int_a^b f(x)dx \tag{2}$$

exists and is finite. This notion is attributed to Kolmogorov, and differs hardly at all from the Q-integral of Titchmarsh [Proc. London Math. Soc. (2) 29 (1928), 49–80]. [For other applications of this notion, see Ul'yanov same Dokl. (N.S.) 90 (1953), 33–36; Mat. Sb. N.S. 35(77) (1954), 469–490; MR 15, 27; 16, 467.] The author states that Kolmogorov proved property (1) for all functions on $[0, 2\pi]$ conjugate to functions in $L_1(0, 2\pi)$ [Fund. Math. 7 (1925), 24–29], but Kolmogorov seems only to have proved that the left side is $o(n^{-1})$. For $f \in L_1(0, 2\pi)$, let $\bar{f}$ denote the conjugate function of f. Theorem: If $f \in L_1(0, 2\pi)$ and if g and $\bar{g}$ are essentially bounded, then

$$(A)\int_0^{2\pi}\bar{f}(x)g(x)dx=-\int_0^{2\pi}f(x)\bar{g}(x)dx.$$

Theorem: If $f \in L_p(0, 2\pi)$ $(p>1)$ and f has period 2π, then

$$(A)\int_0^{2\pi}[f(x+t)-f(x-t)]\tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}t\,dt=-\pi\bar{f}(x)$$

for almost all $x \in [0, 2\pi]$. A formula is also given for inverting the transform $f\to\bar{f}$ $(f \in L_1(-\pi, \pi))$. Proofs are sketched.

E. Hewitt (Princeton, N.J.).

Debrunner, H. Zur Minkowskischen Dimensions- und Massbestimmung beschränkter Punktmengen des euklidischen Raumes. Comment. Math. Helv. 29 (1955), 258–278.

Denote by A_ϱ the ϱ-neighborhood of a non-empty bounded set A in E^k and put successively:

$$g_r(A, \varrho)=|A_\varrho|\varrho^{r-k},$$

where $|A_\varrho|$ is the measure of A_ϱ;

$$g_r'(A)=\liminf_{\varrho\to0+} g_r(A, \varrho),\quad g_r''(A)=\limsup_{\varrho\to0+} g_r(A, \varrho);$$

$$\mu'(A)=\inf r[g_r'(A)=0],\quad \mu''(A)=\inf r[g_r''(A)=0],$$

so that $0\leqq\mu'\leqq\mu''\leqq k$; finally

$$M'(A)=\liminf_{\varrho\to0+} |A_\varrho|\lambda_{k-\mu'}\,\varrho^{\mu'-k},$$

$$M''(A)=\limsup_{\varrho\to0+} |A_\varrho|\lambda_{k-\mu''}\,\varrho^{\mu''-k},$$

where $\lambda_x=\Gamma(1+\tfrac{1}{2}x)\pi^{-\frac{1}{2}x}$.

The numbers $M'(A)$ and $M''(A)$ are the lower and upper Minkowski "Masszahlen" of A. The principal result of the paper is necessary and sufficient conditions for four given numbers to occur as $\mu'(A)$, $M'(A)$, $\mu''(A)$, $M''(A)$ for a suitable set A in E^k. Because there are nine cases, these conditions cannot be enumerated here. There follows a detailed discussion of these set functions for sets on a line.

H. Busemann (Los Angeles, Calif.).

Freilich, Gerald. Two-dimensional measure in 3-space. Proc. Amer. Math. Soc. 6 (1955), 631–633.

The author shows that if A is a plane set and $A\times h$ the Cartesian product of A with a segment of length h, and if C_2^1 and C_3^2 denote Carathéodory measures of dimension 1 in space of dimension 2 and of dimension 2 in space of dimension 3, resp., then $hC_2^1(A)\geqq C_3^2(A\times h)$.

H. G. Eggleston (Cambridge, England).

See also: Cristescu, p. 177.

Functions of a Complex Variable, Generalizations

Denjoy, Arnaud. La quasi-analyticité tirée des séries de fractions rationelles. C. R. Acad. Sci. Paris 240 (1955), 929–932.

S est une série de la forme $\sum A_k/(z-a_k)$. Elle converge (n) (=normalement) sur une ligne s, si: $|A_k|/|z-a_k|<u_k$, indépendant de z sur s, $\sum u_k<\infty$. (E) Si la série S converge (n) sur une ligne s vers une fonction $K(z)$, holomorphe dans un domaine contenant s, les A_k satisfaisant à la condition (L): $\sum|A_k|\log(1/|A_k|)<\infty$, en résulte-t-il que les A_k relatifs aux a_k situés dans un domaine où $K(z)$ est holomorphe sont nuls? Or, on peut, sans restreindre la généralité du problème, imposer aux a_ν des conditions pour qu'il résulte de (L) que $G(u)=\sum A_k e^{a_k u}$ est de la forme $\int_{-\infty}^{\infty}\varphi(t)e^{itu}dt$. C'est ainsi que le problème de l'effectivité des singularités formelles a_k d'une série S, problème (E), a conduit l'auteur à son célèbre théorème sur la quasi-analyticité (la croissance des A_k conditionne celle des maxima des dérivées de $\varphi(t)$).

S. Mandelbrojt (Paris).

Fuksman, N. A. On analytic functions of an integral complex argument. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 159–164. (Russian)

A function $f(z)=\phi(x, y)$ defined at the lattice points $z=m+ni$ (m, n rational integers) is said to be a preanalytic function (p.f.), if

$$\phi(m+1, n+1)-\phi(m, n)=i[\phi(m+1, n)-\phi(m, n+1)]$$

[J. Lelong-Ferrand, Bull. Sci. Math. (2) **68** (1944), 152–180; MR **7**, 149]. If z_1, z_2 are two neighboring lattice points ($|z_1-z_2|=1$), then $\int_{z_1}^{z_2} f(\zeta)\Delta g(\zeta)$ is defined as $\frac{1}{2}[f(z_1)+f(z_2)][g(z_2)-g(z_1)]$. The analogue of Cauchy's Theorem holds for $\int f(\zeta)\Delta\zeta$. Typical examples of p.f.s are: $z^{(0)}\equiv 1$, $z^{(n)}=n\int_0^z \zeta^{(n-1)}\Delta\zeta$; $z^{*(0)}=(-1)^{x+y}-1$; $z^{*(n)}=(-1)^{x+y}4^n z^{(n)}$. The author proves the following. 1) The convolution of two p.f.s $f*g=\int_0^z f(\zeta)\Delta_\zeta g(z-\zeta)$ is a p.f. 2) If the p.f. $g(z)$ has an expansion $\sum c_n z^{(n)}$ or an expansion in the $z^{*(n)}$, then the values of g on the positive real axis determine g completely. He gives a necessary and sufficient condition for the existence of such expansions in terms of the functions $\sum_{m=0}^\infty g(m)t^m$, $\sum_{n=0}^\infty g(in)s^n$. He mentions that the set of p.f.s

$$f_k(x+iy)=e^{2\pi k x i/m}\cot^y\left(\frac{\pi}{4}+\frac{\pi k}{m}\right) \quad (k=0, 1, \cdots, m-1)$$

is a basis of the p.f.s satisfying

$$F(z+m)=F(z), \text{ if } m\not\equiv 0 \pmod 4.$$

W. H. J. Fuchs (Ithaca, N.Y.).

Menger, Karl. The behavior of a complex function at infinity. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 512–513.

The author dislikes the use of $z=t^{-1}$ to describe the behaviour of an $f(z)$ near $z=\infty$. He recommends instead to use a complex analogue of Bouligand's paratingent, and the exterior derivatives

$$E^{k+1}f(a)=\lim [f(z_n')-f(z_n)]/(z_n'-z_n)$$

with $z_n, z_n'\to a$ (when they exist); $E^0 f=f$; $k=0, 1, \cdots$. He gives some terminology from his point of view, including classical conceptions, and discusses advantages of interest. *E. Ullrich* (Giessen).

Wilkoński, A. On the boundedness of root modules of some polynomials. Prace Mat. **1** (1955), 165–168. (Polish. Russian and English summaries)

Set $V(t)=\sum_{i=1}^k (1+A_{k+i+1}(t))\binom{p+1}{i}t^i-1$, where $A_1(t)=0$,

$$A_j(t)=\sum_{i=1}^{j-1} (1+A_{j-1}(t))\binom{p+1}{i}t^i \quad (j=2, 3, \cdots, k).$$

$V(t)=0$ has only one positive root τ. Every polynomial $W(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}+z^n$ of degree $n=p+k$, with $|a_p|=1$, $|a_j|\leqq 1$ for $j=p+1, \cdots, n-1$, has at least k roots $z_1, z_2, \cdots, z_k$ such that $|z_i|\geqq\tau$ ($i=1, 2, \cdots, k$).

S. Mandelbrojt (Paris).

Heinhold, J. Zur Abschätzung der Wurzeln algebraischer Gleichungen. Monatsh. Math. **59** (1955), 203–216.

This paper contains a number of theorems concerning bounds for the absolute values of the roots of algebraic equations with real coefficients as well as criteria for the location of the roots with respect to the unit circle. Well-known results, for example, those of Kakeya [Tôhoku Math. J. **2** (1912), 140–142] and Hurwitz [ibid. **4** (1913), 89–93] for polynomials with positive coefficients, are contained here as special cases. Furthermore, the question of the zeros on a given circle is investigated. One of the theorems given here, for example, is the following: Let $p(z)=p_0-\sum_{\mu=1}^m p_\mu z^\mu$ ($p_0>0$), be a polynomial with real coefficients which has exactly one positive zero r. Let $f(z)=\sum_{v=0}^n a_v z^v$. If the a_v satisfy the conditions

$$\text{(A)} \qquad a_n>0,\ a_0\neq 0,\ \sum_{\mu=1}^m a_{n-\lambda+\mu}p_\mu-a_{n-\lambda}p_0\geqq 0,$$

or

$$\text{(B)} \quad a_0>0,\ a_n\neq 0,\ \sum_{\mu=1}^m a_{\lambda-\mu}p_\mu-a_\lambda p_0\geqq 0 \ (\lambda=1, 2, \cdots, n+m),$$

then the zeros of $f(z)$ satisfy the conditions $|z|\leqq 1/r$ or $|z|\geqq r$, respectively. (Here $a_v=0$ for $v<0$ or $v>n$.)

E. Frank (Chicago, Ill.).

Dvoretzky, A., and Erdös, P. On power series diverging everywhere on the circle of convergence. Michigan Math. J. **3** (1955), 31–35.

The first example of a power series $\sum a_n z^n$ (with $\lim a_n=0$) which diverges everywhere on the unit circle was given by N. Lusin [Rend. Circ. Mat. Palermo **32** (1911), 386–390]. Recently, the reviewer gave an example of this kind with non-negative a_n [Michigan Math. J. **2**, 175–177 (1955); MR **16**, 578]. The principal result of the present paper is the following theorem, which allows a very general construction of power series of Lusin's type and contains the reviewer's example as a special case. Let $\{b_n\}$ be a sequence of complex numbers with $|b_n|\geqq|b_{n+1}|$ and $\sum|b_n|^2=\infty$. Then there exists a sequence $\{\varepsilon_n\}$ of 0's and 1's such that the power series $\sum\varepsilon_n b_n z^n$ diverges everywhere on the unit circle. Several extensions of this theorem are given. It is, for instance, possible under the same conditions to choose the ε_n so that $\sum\varepsilon_n b_n z^n$ has unbounded partial sums everywhere on the unit circle. *F. Herzog* (East Lansing, Mich.).

Rivlin, T. J. On sufficient conditions for overconvergence. Proc. Amer. Math. Soc. **6** (1955), 597–602.

Let $\sum a_n z^n$ be a power series with radius of convergence 1 and with gaps defined by the relation $a_n=0$ for $m_k<n<n_k$. The polynomial between the $(k-1)$st gap and the kth gap may be written in the form $z^{n_{k-1}}p_k(z)$. The author proves several theorems giving sufficient conditions for overconvergence of the power series. These theorems are of two different types.

The first type is examplified by the following statement. If $\limsup (m_k-n_{k-1})/n_{k-1}>0$, and if $\limsup |a_n|^{1/n}<1$, as $n\to\infty$ through the values n_k, and if the roots of the polynomials $p_k(z)$ satisfy certain conditions, then the sequence of the m_kth partial sums of the power series converges uniformly in a neighborhood of the point $z=1$. The conditions on the roots of the $p_k(z)$ are too complicated to be given here in detail; they restrict these roots to certain regions of the complex plane and involve the values of the two superior limits mentioned above.

An example of the second type of theorem is the following. Under conditions similar to the ones indicated above, it is possible, for each point $e^{i\phi}$ on the unit circle, to find a sequence of arguments ϕ_n, depending on ϕ, such that the power series $\sum a_n e^{i\phi_n}z^n$ overconverges in a neighborhood of the point $e^{i\phi}$. *F. Herzog.*

Ricci, Giovanni. Sui punti critici degli elementi analitici. Matematiche, Catania **9** (1954), 43–81.

L'auteur cherche à systématiser quelques résultats classiques sur la distribution des singularités d'une série de Taylor sur son cercle de convergence. *S. Mandelbrojt.*

Ricci, Giovanni. Sull'insieme delle serie di potenze non prolungabili. Giorn. Mat. Battaglini (5) 3(83) (1955), 5–11.

Let two Taylor series $\sum a_n z^n$ and $\sum b_n z^n$ be regarded as equivalent provided the radius of convergence of the series $\sum (a_n - b_n) z^n$ is greater than 1; and let T be the space of equivalence classes $[a]$ of Taylor series that converge in $|z|<1$. If $\varepsilon = \{\varepsilon_n\}$ is a positive sequence, with $\varepsilon_n^{1/n} \to 1$, the ε-neighborhood of a point $[a]$ in T is defined as the set of equivalence classes $[x]$ which have the property that, for some series $\sum x_n z^n$ and $\sum a_n z^n$ in $[x]$ and $[a]$, the condition $|x_n - a_n| < \varepsilon_n$ $(n=0, 1, \cdots)$ holds. If S is a closed set on $|z|=1$, let $\mathfrak{M}(S)$ denote the set of those points in T whose corresponding Taylor series cannot be continued analytically across the unit circle at any point of S. It is known that $\mathfrak{M}(S)$ is open [Pólya, Acta Math. **41** (1917), 99–118]. The author presents a remarkably simple and elementary new proof.

He chooses a denumerable set $\{e^{i\phi_s}\}$ whose closure is the set S, and he uses the linear transformations $z = e^{i\phi_s} Z/(1-Z)$. If $\sum a_n z^n$ is a Taylor series in the equivalence class $[a]$ in $\mathfrak{M}(S)$, then each of the functions

$$F(Z, \phi_s) = (1 + z e^{-i\phi_s}) \sum a_n z^n$$

is regular in the half-plane $\Re Z < 1/2$ and is therefore represented by a Taylor series $\sum A_k(\phi_s) Z^k$ with $\limsup |A_k(\phi_s)|^{1/k} = 2$. This permits the selection of two sequences $\{s_m\}$ and $\{k_m\}$ of positive integers such that each positive integer occurs infinitely often in $\{s_m\}$, and such that $k_{m+1} > 4k_m$ and

$$|A_{k_m}(\phi_{s_m})|^{1/k_m} \to 2.$$

A sequence $\{\varepsilon_n\}$ is then chosen such that $0 < \varepsilon_n \leqq 1$, $\varepsilon_n^{1/n} \to 1$, and such that $2^{k_m} \varepsilon_n$ is small compared with A_{k_m} for $k_m/4 \leqq n \leqq k_m$. The resulting ε-neighborhood of $[a]$ lies in $\mathfrak{M}(S)$. *G. Piranian* (Ann Arbor, Mich.).

Ricci, Giovanni. Variazioni di segno condizionate e teorema di Fabry. Ann. Mat. Pura Appl. (4) 38 (1955), 1–31.

Verfasser betrachtet Potenzreihen $\sum a_n z^n$ mit Konvergenzradius 1 und fragt nach hinreichenden Bedingungen für (iii) „Der Punkt 1 ist singulär" [vgl. E. Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, 2. Aufl., Springer, Berlin, 1929, pp. 14–16, 72–86]. Nach Pringsheim gilt (iii), wenn $a_n \geqq 0$ ist. Statt dessen kann man verlangen, daß die a_m nur wenig Zeichenwechsel haben, sogar mit Beschränkung auf gewisse Intervalle I_h $(h=1, 2, \cdots)$ der Form

$$(1-\theta) n_h \leqq m \leqq (1+\theta) n_h$$

(wo $n_h \to \infty$, θ fest, $0<\theta<1$). Dabei muß noch gefordert werden, daß die zu den I_h gehörigen Stücke der Potenzreihe schon allein den Radius 1 ergeben. Damit kommen wir zu den Fabryschen Bedingungen für (iii), wobei γ_h eine geeignet gewählte Folge ist: (ii) $\mathrm{Re}\,(a_{n_h} e^{-i\gamma_h}) > 0$, $\mathrm{Re}\,(a_{n_h} e^{-i\gamma_h})^{1/n_h} \to 1$ für $h \to \infty$, (i) „Die Zahl der Zeichenwechsel (Nullen werden weggelassen) von $\mathrm{Re}\,(a_m e^{-i\gamma_h})$ in I_h ist $o(n_h)$". Aus (i) und (ii) folgt also (iii). Die Ungleichung in (ii) dient nur der Normierung und ist für das folgende bequem. Verfasser [Boll. Un. Mat. Ital. (3) **9** (1954), 126–135; MR **16**, 24] hat die Bedingung (i) ersetzt durch (i') $\mathrm{Re}\,(a_m e^{-i\gamma_h}) \geqq -K \exp(u^2/n_h) \cdot \mathrm{Re}\,(a_{n_h} e^{-i\gamma_h})/n_h$ für m in I_h, wobei $u = |n_h - m|$ gesetzt ist (siehe Satz I der Arbeit). Diese Bedingung läßt auch mehr als $o(n_h)$ Zeichenwechsel zu, solange sich diese im Rahmen der Ungleichung (i') abspielen. Dies führt Verfasser zur Definition der ψ-Variation eines Zahlen-N-tupels u_m bezüglich einer Funktion ψ: Hierbei werden die Glieder mit $u_m \leqq \psi(m)$ Null gesetzt und die Zeichenwechsel des entstehenden N-tupels abgezählt. Nun (Satz II) läßt sich (i) auch ersetzen durch (i'') „Die ψ_h-Variation von $\mathrm{Re}\,(a_m e^{-i\gamma_h})$ in I_h ist $\leqq V_h$, wobei $V_h \to \infty$ eine geeignete Folge mit $V_h = o(n_h)$ darstellt,

$$\psi_h(u) = K \exp\{u^2/n_h) - \Phi(u, V_h)\} \cdot \mathrm{Re}\,(a_{n_h} e^{-i\gamma_h})/(n_h V_h^2)$$

und $\Phi(u, v) = u\{\log(v/u) + 4\}$ für $1 \leqq u \leqq v$ sowie

$$\Phi(u, v) = v\{\log(u/v) + 4\}$$

für $1 \leqq v \leqq u$ gesetzt wird und wiederum $u = |n_h - m|$ gilt." Im Sonderfall V_h=constans genügt die Forderung, daß die ψ_h-Variation $\leqq V_1 - 1$ ist. Der Beweis ist ähnlich wie der bei Landau, loc. cit. Ein Beispiel zeigt, daß Satz II umfassender als der Fabrysche ist. Aus Satz II folgen die Sätze III–V, bei denen in der Bedingung (i) die I_h durch kleinere Intervalle I_h^* der Länge $o(n_h)$ ersetzt werden, dafür aber die Zahl der Zeichenwechsel schärfer eingeschränkt wird. Die Sätze II–V liefern Nichtfortsetzbarkeitskriterien (Satz VI), von denen eines einer Bedingung von F. Lösch [Math. Z. **32** (1930), 415–421] ähnelt.

K. Zeller (Tübingen).

San Juan Llosá, Ricardo. The foundations of a general theory of divergent series. Rev. Acad. Ci. Madrid **44** (1950), 223–232; **45** (1951), 121–149, 241–254, 333–350; **46** (1952), 257–271, 383–439. (Spanish)

[This is the first part of the memoir of which a summary appeared in Univ. Lisboa. Revista Fac. Ci. A. (2) **2** (1951), 45–76; MR **14**, 1077. For the significance of the title cf. Acta Math. **75** (1943), 247–254; Mem. Real. Acad. Ci. Madrid. Ser. Ci. Exact. **2** (1942); MR **7**, 8; **14**, 543.] Except in the introduction, the paper under review is concerned with entire functions and asymptotic expansions. The first chapter deals with connections between the maximum modulus and the coefficients of an entire function whose growth can be compared with that of some iterated exponential. [Some of this work has been extended by Azpeitia, Rev. Mat. Hisp.-Amer. (4) **14** (1954), 3–25, 83–103, 179–193, 221–236; MR **16**, 915; and Sales Vallés, Collect. Math. **7** (1954), 69–94; MR **16**, 1094.] Let $\log_p r$ denote $\log \cdots \log r$ with p log signs and let $e_q(r)$ denote $\exp \cdots \exp(r)$ with q exp signs. An entire function is of ∞^p-order $\varrho < \infty$ if $\limsup \{\log_{p+2} M(r)\}/\log r = \varrho$; of regular growth when the corresponding limit exists. It is of perfectly regular growth of ∞^p-order ϱ, type σ, if $\lim r^{-\varrho} \log_{p+1} M(r) = \sigma$ $(0<\sigma<\infty)$. More generally, a function of ∞^{p+q-1}-order ϱ has perfectly regular $\infty_q{}^p$-growth of type σ if $\lim \{\log_p M(r)\}/e_q(r^\varrho) = \sigma$. Necessary and sufficient conditions for these kinds of growth are given in terms of the coefficients. Most of the results (expressed in different terminology) are said to have been given without proof by Lindelöf, except for the case of perfectly regular $\infty_q{}^q$-growth.

The second chapter is devoted to asymptotic power series and in particular to the asymptotic series for integrals $\int \alpha(t) \Phi(tz)\,dt$ with integration along a ray in the complex plane. There are particularly detailed treatments of the Laplace transform, and of the Mittag-Leffler transform in which $\Phi(t) = \exp(-t^{1/\alpha}) t^{(1/\alpha)-1}$. *R. P. Boas, Jr.*

Mironov, V. T. On a class of rational interpolation series. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 215–218. (Russian)

Detailed discussion of the convergence properties of the 'q-series' $\sum c_n \pi_n(z)$, $\pi_n(z) = \prod_{k=1}^n (1 - z/uk^{1/q})/(1 - z/vk^{1/q})$.

It is assumed that $|u/v|^q$ is irrational. Principal results are: For $q\geq 1$, every q-series has a half-plane of convergence $\Re(u^{-1}-v^{-1})z>\lambda$ and a half-plane of absolute convergence. If $q<1$, then convergence (absolute convergence) at one point z_0 distinct from the zeros of the π_n implies convergence (absolute convergence) everywhere, except at the poles of the π_n. The sum of a q-series is meromorphic with simple poles at the points $vn^{1/q}$ in the region of convergence. Estimates for the order of magnitude of a q-series are given.

W. H. J. Fuchs (Ithaca, N.Y.).

Gavrilov, L. I. On K-extension of polynomials. Mat. Sb. N.S. 36(78) (1955), 271–274. (Russian)

Let $f(z)=1+a_1z+\cdots+a_nz^n$ be any polynomial and let C be a closed Jordan curve containing the origin and piecewise regular; then it is possible to find a polynomial $\phi(z)=b_1z^{n+1}+\cdots+b_mz^m$ such that $f(z)+\phi(z)$ has all its zeros on C. Results of this kind were previously given by the author [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 32 (1941), 235–236; MR 3, 236] and Tchebotaröw [ibid. 32 (1941), 3–6; MR 3, 236].

S. Mandelbrojt (Paris).

Have

★ **Buck, R. C. Expansion theorems for analytic functions. I.** Lectures on functions of a complex variable, pp. 409–419. The University of Michigan Press, Ann Arbor, 1955. $10.00.

The author treats the problem of polynomial expansions of analytic functions as a problem of functional analysis in locally convex, topological linear spaces. Let E be such a space, $\pi=(p_0, p_1, \cdots)$ an enumerable sequence of elements of E. $E(\pi)$ is the set of x which have a convergent expansion (1) $x=\sum c_np_n$. If the subset A of E satisfies $A\subset E(\pi)$, then π is effective for A. If T is a continuous linear transformation of E, then $\{T(p_n)\}$ is effective for $T(A)$. In particular, if $T=I+S$ (I identity transformation, $\|S\|<1$), then T^{-1} is a linear operator and T carries bases of E into bases. This observation proves the 'closeness theorems' of Paley-Wiener et al.

The author then specializes E to the space E_0 of entire functions with the topology of uniform convergence on compact sets. He proves the theorem: All continuous linear transformations of the space E_0 into itself are of the form $T=\sum a_n(z)/n!(d/dz)^n=\phi(z; d/dz)$, where $a_n(z)\in E_0$ and $\lim\sup_{n\to 0}(M(r, a_n))^{1/n}<\infty$ for each r; $M(r, a_n)=\sup_{|z|\leq r}|a_n(z)|$. If the transformation T transforms polynomials into polynomials, then the $a_n(z)$ are polynomials.

For functions of finite exponential type $T(f)$ can be written as an integral involving the Borel-transform $F(w)$ of $f(z)$ [if $f(z)=\sum c_nz^n$, $F(w)=\sum c_n/n!w^{n+1}$]. In fact

$$T(f)(z)=\frac{1}{2\pi i}\int_\Gamma e^{wz}\phi(z, w)F(w)dw$$

for a suitable closed contour Γ. This representation is generalized by replacing the exponential function by other suitable comparison functions. An expansion $e^{wz}\phi(z; w)=\sum p_n(z)u_n(w)$ yields an expansion

$$T(f)(z)=\sum c_np_n(z),\quad c_n=\frac{1}{2\pi i}\int u_n(w)F(w)dw.$$

Finally the following theorem is proved. Let S be the additive transformation defined by $S(z^n)=z^n/n!$. If T is a continuous linear transformation of E_0 such that $T^{-1}S$ is a continuous linear transformation from the space of polynomials, considered as a sub-space of functions regular in $|z|<R$ into the sub-space of polynomials in E_0, then $\{T(z^k)\}_{k=0}^\infty$ is effective for exponential functions of type $<1/R$. A generalization to other classes of entire functions defined in terms of comparison functions other than the exponential function is possible.

W. H. J. Fuchs (Ithaca, N.Y.).

Ahmad, Mansoor. Cauchy's theorem and its converse. Acta Math. 93 (1955), 15–25.

Let C be a contour in the z-plane with the property that there exists a point z_0 within C such that every ray from z_0 intersects C in precisely one point. The author considers the family of rays through z_0 and the family of curves C_λ, $0<\lambda<1$, defined by the equation $\zeta=z_0+\lambda(z-z_0)$, $z\in C$. Let $f(z)$ be a single-valued function in the domain bounded by C. By imposing suitable continuity and differentiability assumptions on $f(z)$ along each of the two families of curves, the author obtains a sufficient condition for the validity of the equation $\int_C f(\zeta)d\zeta=0$ and of Cauchy's integral formula. A similar generalization of Morera's theorem is also obtained. Extensions to domains other than stars are indicated.

W. Seidel (Notre Dame, Ind.).

Lax, Peter D. Reciprocal extremal problems in function theory. Comm. Pure Appl. Math. 8 (1955), 437–453.

Let D be an n-fold connected domain in the complex plane with boundary curves C which, apart from a finite number of corners, have continuous tangents. D' is a closed subset of D, and $\Phi(x, y)$ is continuous on D'. The important problem is to maximize $|L(f)|$, where

$$L(f)=\iint_{D'} f(z)\Phi(x, y)dx\,dy \quad (z=x+iy),$$

amongst all $f(z)$, regular and with $|f|\leq 1$ in D, that have zeros of given order n_i at a finite number of given points z_i in D. Many classical problems are special cases of this general set up. By Cauchy's formula L can be written as $L(f)=\int_{C'} f(w)k(w)dw$ with the kernel

$$k(w)=(2\pi i)^{-1}\iint_{D'}(w-z)^{-1}\Phi(x, y)dx\,dy,$$

where C' are n curves in D surrounding D'. Now the following abstract principle is applied: Let B be a (complex) linear normal space, k an element of it, and S a subspace. Then $\max|l(k)|=\inf_{s\in S}\|k-s\|$, the distance of k from S, where the max refers to all bounded linear functionals l with $\|l\|=1$ that vanish on S. [This principle is dual to one used for similar problems by the reviewer and H. S. Shapiro, Acta Math. 90 (1953), 287–318; MR 15, 516; both principles are easy interpretations of the Hahn-Banach extension theorem in the complex case.] The principle is applied to B, the space of continuous functions on C with L_1-norm; k is the above kernel on C; and S is the subspace of all $s(z)$ on C, where $s(z)$ is regular in $D+S$ except for poles of order at most n_i at the z_i. The results are as follows: First,

$$\max|L(f)|=\min\int_C|k(w)-g(w)|\,|dw|,$$

the min taken with respect to all $y(z)$ regular in D with poles of order at most n_i at the z_i. The maximal $f_0(z)$ is unique, continuous in $D+C$, and $|f_0(w)|=1$ on C. Every minimal of $g_0(z)$ is continuous in $D+C$, and on C, $[k(w)-g_0(w)]f_0(w)dw\geq 0$. The latter relation allows one, in certain cases, to determine the mapping degree of the map by f_0 of D onto $|w|<1$. This important paper is somewhat marred by numerous misprints and a rather loose presentation.

W. W. Rogosinski.

Nehari, Zeev. An integral equation associated with a function-theoretic extremal problem. J. Analyse Math. 4 (1955), 29–48.

Let $L^2(D)$ be the family of functions, single-valued and regular in a finite domain D and square-integrable over the boundary C which consists of a finite number of rectifiable Jordan curves. The starting problem is to minimize $\int_C |f(z)|^2 ds$, $ds=|dz|$, for $f(z)\in L^2(D)$ under the condition that $\int_{C_1}|f(z)|^2 ds=1$, where C_1 is an assigned subset of D consisting of a finite number of rectifiable Jordan arcs or curves. It is shown that $\int_C |g(z)|^2 ds \geqq \lambda_1$ for any admissible $g(z)$, where λ_1 (>0) is the lowest eigenvalue of the integral equation

$$f(\zeta)=\lambda\int_{C_1} K(z,\zeta)f(z)ds,$$

$K(z,\zeta)$ being the Szegö kernel function of $L^2(D)$; the equality can appear only if $g(z)$ is a solution of the integral equation associated with $\lambda=\lambda_1$. On the other hand, the author proves the following result: Let ζ_ν ($\nu=1,\cdots,n$) be any points in a subdomain D_1 bounded by C_1 of D; then

$$\lambda_1\sum_{\nu,\mu=1}^{n} a_\nu\bar{a}_\mu K(\zeta_\mu,\zeta_\nu)\leqq\sum_{\nu,\mu=1}^{n} a_\nu\bar{a}_\mu K_1(\zeta_\mu,\zeta_\nu)\quad(n=1,2,\cdots),$$

where $K_1(z,\zeta)$ is the kernel associated with D_1 and the a's are indeterminate variables; every inequality is sharp. Analogous problems for the Bergman kernel function are discussed.

Y. Komatu (Tokyo).

Džrbašyan, M. M. Estimates for the derivatives of entire functions of finite order and normal type. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **8** (1955), no. 2, 1–16. (Russian. Armenian summary)

The author establishes inequalities for the derivatives of entire functions that are bounded in angular regions. In the first theorem the function is of order $\varrho\geqq\frac{1}{2}$, mean type σ, and bounded by M for $|\arg z|\geqq\frac{1}{2}\pi/\varrho$; in the second, it is of order $\varrho\geqq1$ and mean type σ, and bounded by M for $|\arg z|$ and $|\arg z+\pi|\leqq\frac{1}{2}\pi(1-\varrho^{-1})$. In each case bounds (not sharp) are given for $|f^{(k)}(z)|$ on the boundary. In particular, in both cases $|f^{(k)}(0)|\leqq k!(\varrho/k)^{k/\varrho}C(k,\varrho)$, with $C(k,\varrho)=\frac{1}{2}(2-\varrho^{-1}+\varrho^{-1}e^{k/\varrho})$ in the first case and $1-\varrho^{-1}+\varrho^{-1}e^{k/\varrho}$ in the second; this is slightly more than follows from the usual estimate for the coefficients in terms of the order and type with no boundedness condition.

R. P. Boas, Jr. (Evanston, Ill.).

Boas, R. P., Jr. Moments of analytic functions. Proc. Amer. Math. Soc. **6** (1955), 412–413.

The author proves the following: If $f(z)$ is holomorphic and of exponential type for $x\geqq0$, and is bounded on the imaginary axis, and if for some θ and for an infinity of n

$$\int_0^\infty |f(re^{i\theta})|r^n dr<n^n e^{-n\varphi(n)},\quad \varphi(n)\to\infty,$$

then $f(z)\equiv0$. This improves Sunyer Balaguer's theorem [Rev. Mat. Hisp.-Amer. (4) **13** (1953), 241–246; MR **15**, 613] in which $\varphi(n)=\alpha\log n$ ($0<\alpha<1$). *S. Mandelbrojt.*

Shah, Tao-Shing. On the coefficients of schlicht functions. J. Chinese Math. Soc. (N.S.) **1** (1951), 98–107. (Chinese summary)

Let $f(z)=\sum_{n=1}^{\infty} a_n z^n$ be regular and univalent in $|z|<1$. The author obtains the following estimates: $|a_5|<5.5079$, $|a_6|<7.0542$, $|a_7|<8.8494$, $|a_8|<10.8408$, $|a_9|<13.0785$, $|a_{10}|<15.2995$, $|a_{11}|<18.0159$.

Further, the author finds all such functions for which all the coefficients are algebraic integers of an imaginary quadratic field. There are 45 such functions, all of the form $z/(1+b_1z+b_2z^2)$. The list given by the author is not consistent with that given by V. Linis [Amer. Math. Monthly **62** (1955), 109–110; MR **16**, 809] who solved the problem for Gaussian integers and found 15 such functions. Apparently the latter author omitted $z/(1\pm iz^2)$, $z/(1\pm(1+i)z+iz^2)$, and $z/(1\pm(1-i)z-iz^2)$.

A. W. Goodman (Lexington, Ky.).

Shah, Tao-Shing. The principle of area in the theory of univalent functions. Acta Math. Sinica 3 (1953), 208–212. (Chinese. English summary)

Let $F(\zeta)=\zeta+\sum_{n=0}^{\infty}\alpha_n\zeta^{-n}$ be regular in G: $1<|\zeta|<\infty$. Let $Q_m(t)$ be a polynomial of degree m, and set $Q_m(F(\zeta))=\sum_{n=-m}^{\infty}C_n\zeta^{-n}$. Wolibner [Colloq. Math. **2**, 249–253 (1952); MR **14**, 35] proved that if

$$\text{(1)}\qquad \sum_{n=-m}^{\infty} n|C_n|^2\leqq0$$

for every polynomial $Q_m(t)$ then $F(\zeta)$ is univalent in G. The necessity of the condition (1) was proved by Goluzin [Mat. Sb. N.S. **8(50)** (1940), 277–284; MR **2**, 185] and Biernacki [Bull. Sci. Math. (2) **70** (1946), 51–76; MR **8**, 326]. The author now shows by an area principle that if (1) is satisfied then

$$\text{(2)}\quad \left|\sum_{\mu,\nu=1}^{n}\gamma_\mu\gamma_\nu\log\frac{F(\zeta_\mu)-F(\zeta_\nu)}{\zeta_\mu-\zeta_\nu}\right|\leqq-\sum_{\mu,\nu=1}^{n}\gamma_\mu\gamma_\nu\log\left(1-\frac{1}{\zeta_\mu\bar{\zeta}_\nu}\right),$$

and

$$\text{(3)}\qquad \left|\sum_{m,n=1}^{\infty}d_{mn}\lambda_m\lambda_n\right|\leqq\sum_{m=1}^{\infty}\frac{|\lambda_m|^2}{m},$$

where γ_μ, ζ_μ, λ_m are arbitrary, $|\zeta_\mu|>1$, and d_{mn} is defined by

$$\log\frac{F(\zeta_1)-F(\zeta_2)}{\zeta_1-\zeta_2}=\sum_{m,n=1}^{\infty}d_{mn}\frac{1}{\zeta_1{}^m\zeta_2{}^n}.$$

This gives an elementary proof of (2) and (3) for univalent functions. (Reviewed from the author's summary.)

A. W. Goodman (Lexington, Ky.).

Shah, Tao-Shing. On the product of mapping radii for a system of non-overlapping domains. Acta Math. Sinica 3 (1953), 1–7. (Chinese and English)

Let $a_1, a_2, \cdots, a_n$ be n distinct fixed points and let $f_k(\zeta)$ be regular and univalent in $|\zeta|<1$, $f_k(0)=a_k$ ($k=1,2,\cdots,n$). Let B_k be the image of $|\zeta|<1$ under $f_k(\zeta)$, and suppose the regions B_k are pairwise disjoint. Goluzin [Mat. Sb. N.S. **29(71)** (1951), 455–468; MR **13**, 454] derived a system of differential equations satisfied by the set of functions which maximizes $I=\prod_{k=1}^{n}|f_k'(0)|$.

The author now proves that the functions of the extremalizing set are all branches of one analytic function $z=f(\zeta)$ which in $|\zeta|<1$ satisfies the differential equation.

$$\left(\frac{dz}{d\zeta}\right)^2\left(Q(z)\prod_{\nu=1}^{n}(z-a_\nu)+\sum_{k=1}^{n}\prod_{\nu=1,\,\nu\neq k}^{n}(z-a_\nu)(a_k-a_\nu)\right)=\frac{1}{\zeta^2}\prod_{\nu=1}^{n}(z-a\ )^2$$

where $Q(z)$ is a polynomial of degree less than $n-2$. (Reviewed from the author's summary.)

A. W. Goodman (Lexington, Ky.).

Shah, Tao-Shing. The product of the mapping radii of non-overlapping domains. Acta Math. Sinica 5 (1955), 27–36. (Chinese. English summary)

The author continues his work on the problem described in the preceding review. Set

$$g(z)=\int_{\infty}^{z}(P(z))^{\frac{1}{2}}\prod_{k=1}^{n}(z-a_k)^{-1}dz.$$

It is proved that if $P(z)$ is any polynomial of degree less than $2n-2$ such that $P(a_k)=\prod_{\nu=1,\nu\neq k}^{n}(a_\nu-a_k)^2$ and $\Re g(z_k)=0$ for each zero z_k of $P(z)$, then the regions with boundaries defined by $\Re g(z)=0$ are extremal for I. Further, the corresponding mapping functions are the inverse functions of $\zeta=e^{g(z)}$.

A similar result is obtained for the following problem. Let G be a region which contains 0 but not the points $a_1, a_2, \cdots, a_n$ and let $f(\zeta)$ map $|\zeta|<1$ conformally onto G, with $f(0)=0$. Find the maximum of $|f'(0)|$. (Reviewed from the author's summary.) *A. W. Goodman.*

Kung Sun. Some theorems on symmetric schlicht functions. Acta Math. Sinica 3 (1953), 251–260. (Chinese. English summary)

Let $f_k(z)=z+\sum_{n=1}^{\infty} a_{nk+1}{}^{(k)}z^{nk+1}$ be regular and univalent in $|z|<1$. The author proves a number of results. A typical one is $|a_n{}^{(3)}|<6.10\cdots/n^{\frac{1}{3}}$. (Reviewed from the author's summary.) *A. W. Goodman.*

Gun Syn [Kung Sun]. On coefficients of univalent functions. Acta Math. Sinica 4 (1954), 87–103. (Chinese. Russian summary)

With the notations of the preceding review the author proves that

$$|a_n{}^{(1)}|<\tfrac{1}{2}n\exp\{1-\tfrac{1}{2}(2-a)^2\}+O(n^{1/2}),$$

where $a=|a_2{}^{(1)}|$. Further he proves that

$$|a_n{}^{(2)}|<2^{-\frac{1}{2}}3^{\frac{1}{2}}\exp\{\tfrac{1}{2}-(1-b)^2\}+o(1),$$

where $b=|a_3{}^{(2)}|$. Similarly $n^{\frac{1}{3}}|a_n{}^{(3)}|$ is bounded in terms of $|a_4{}^{(3)}|$. (Reviewed from the author's summary.)

A. W. Goodman (Lexington, Ky.).

Kung Sun. Distortion theorems and coefficients of schlicht functions. Acta Math. Sinica 3 (1953), 231–250. (Chinese. English summary)

Let $f(z)=z+\sum_{n=2}^{\infty} c_n z^n$ be regular and univalent in $|z|<1$. The author obtains bounds for $|f(z)|$ and $|f'(z)|$ in several forms as functions of c_2 and $|z|=r$. The formulas are too complicated to be reproduced here, but the simplest among them are $|f(z)|\leqq r(1-r)^{-2}e^{-q}$ and $|f'(z)|\leqq(1+r)(1-r)^{-3}e^{-2q}$, where $q=(1+r)^2r(2-|c_2|)^2/8$. A similar result is obtained for the special case $c_2=c_3=0$. These bounds imply new estimates for $|c_n|$, for example if $c_2=c_3=0$, then $|c_n|\leqq n/e^3+O(\log n)$. (Reviewed from the author's summary.) *A. W. Goodman.*

Kung Sun. The sections of schlicht functions. Acta Math. Sinica 4 (1954), 105–112. (Chinese. English summary)

Let $f_k(z)=z+\sum_{n=1}^{\infty} a_n z^{nk+1}$ be regular and univalent in $|z|<1$, and let $\sigma_n{}^{(k)}(z)$ be the sum of the first $n+1$ terms. It is proved that $\sigma_n{}^{(3)}(z)$ is univalent in $|z|^3<3/8$, and this result is sharp. The same theorem was proved independently by Iliev [Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 621–622; MR **16**, 809]. Combined with earlier results, the conjecture that $\sigma_n{}^{(k)}(z)$ is univalent in $|z|^k<k/2(k+1)$ is now known to be true for $k=1, 2, 3$. A similar result is proved for the partial sums of $F(\zeta)=\zeta+\sum_{n=1}^{\infty}\alpha_n\zeta^{-n}$, regular and univalent in $1<|\zeta|<\infty$. (Reviewed from the author's summary.) *A.W. Goodman.*

Kung Sun. On mean valent functions. Acta Math. Sinica 4 (1954), 245–257. (Chinese. English summary)

Let $f(z)=\sum_{n=p}^{\infty} c_n z^n$ be mean p-valent as defined by Littlewood [Proc. London Math. Soc. (2) 23 (1925), 481–519] and let $M(r)=\max_{|z|=r}|f(z)|$. The author proves that for $\lambda>0$

$$\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\leqq 2\pi\lambda p\int_0^r M^\lambda(\varrho)\varrho^{-1}d\varrho$$

with equality only if $f(z)=az^p$. If $f(z)$ is mean 1-valent then $f(z)$ is univalent in $|z|<r_0$ where r_0 is the positive root of $1-r^2-r^3=0$. (Reviewed from the author's summary.) *A. W. Goodman* (Lexington, Ky.).

Kung Sun. The function $k(t)$ in Golusin and Löwner's differential equation. Acta Math. Sinica 3 (1953), 225–230. (Chinese. English summary)

Löwner [Math. Ann. **89** (1923), 103–121] showed that if $f(z)$ belongs to a certain class of functions univalent in $|z|<1$ then $f(z)=\lim_{t\to\infty} e^t f(z, t)$, where $f(z, r)$ satisfies the equation

$$\frac{\partial f}{\partial t}=-f\frac{1+k(t)f}{1-k(t)f}\quad(0\leqq t,\ |z|<1)$$

with $f(z, 0)=z$ and $k(t)$ is a piecewise continuous function with $|k(t)|=1$. The author now shows that if $k(t)=e^{i\theta}p(t)$, where θ is a real constant and $p(t)$ is real then $f(z)=z/(1-ce^{i\theta}z+e^{2i\theta}z^2)$ with $|c|\leqq 2$. (Reviewed from the author's summary.) *A. W. Goodman* (Lexington, Ky.).

Hu Ke. On the distortion of schlicht functions. Acta Math. Sinica 4 (1954), 259–262. (Chinese. English summary)

Let $F(\zeta)=\zeta+\sum_{n=1}^{\infty}\alpha_{2n}\zeta^{1-2n}$ be regular and univalent in $1<|\zeta|<\infty$. Bounds are obtained for certain forms involving the difference quotients of $F(\zeta)$. The results are too complicated to be reproduced here. (Reviewed from the author's summary.) *A. W. Goodman.*

Čin, Yan'-Šin. On the arguments of the coefficients in the expansion of a univalent function. Acta Math. Sinica 4 (1954), 81–86. (Chinese. Russian summary)

Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular in $|z|<1$. Suppose that for some fixed N, $|a_N|>1/N$. Then there are real numbers φ_1 and φ_2 such that both of the functions

$$g_1(z)=z+e^{i\varphi_1}\sum_{n=2}^{\infty}a_n z^n,$$

$$g_2(z)=z+\sum_{n=2}^{N-1}a_n z^n+e^{i\varphi_2}a_N z^N+\sum_{n=N+1}^{\infty}a_n z^n$$

are not univalent. These results are sharp.

A. W. Goodman (Lexington, Ky.).

Hayman, W. K. The asymptotic behaviour of p-valent functions. Proc. London Math. Soc. (3) **5** (1955), 257–284.

Let Δ denote either the circle $|z|<1$ or an annulus $r_0<|z|<1$, and let p be a positive real number. The author calls $f(z)$ mean p-valent in Δ if (i) $f(z)$ may be analytically continued without singularity throughout Δ so that $|f(z)|$ remains single-valued, and (ii) $p(R)\leqq p$, $0<R<\infty$, where

$$2\pi p(R)=\int_0^{2\pi}n(Re^{i\phi})d\phi,$$

and where $n(w)$ is the total number of roots of $f(z)=w$ in Δ for a fixed branch $f(z)$ (Δ cut along a radius if necessary). This definition is somewhat less general than that of D. C. Spencer [Trans. Amer. Math. Soc. **48** (1940), 418–435; MR **2**, 82], where only

$$\int_0^\varrho p(R)dR^2 \leqq p\varrho^2,\ 0<\varrho<\infty,$$

but the class of mean p-valent functions so defined includes the functions $f(z)$ p-valent in Δ.

The main result is that if

$$\text{(A)}\qquad f(z)=z^\mu \sum_{-\infty}^{\infty} a_{kn}z^{kn}$$

is mean p-valent in $r_0<|z|<1$, then the limit

$$\alpha=\lim_{r\to 1}(1-r^k)^{2p/k}\cdot \max_{|z|=r}|f(z)|$$

exists finitely, and if $1\leqq k<4p$, then

$$\lim_{n\to+\infty}\frac{|a_{kn}|}{n^{2p/k-1}}=\frac{\alpha}{\Gamma(2p/k)}.$$

In case

$$f(z)=z^p(1+a_kz^k+a_{2k}z^{2k}+\cdots)$$

is mean p-valent in $0<|z|<1$, we have $\alpha<1$ unless $f(z)\equiv z^p(1-z^ke^{i\theta})^{-2p/k}$. For $p=k=1$ this includes a proof of the conjecture $|a_n|\leqq n+1$ for all sufficiently large n, and for $p=1$, $k=2$ also gives $|a_{2n}|\leqq 1$ for all sufficiently large n and fixed $f(z)$.

A special study is made of those functions $f(z)$ given by (A) with $k=1$ where $\alpha>0$. It is shown in this case that the functions attain their maximum modulus in the immediate neighborhood of a single radius and are small elsewhere. *M. S. Robertson* (New Brunswick, N.J.).

Kryż, J. On the maximum modulus of univalent functions. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 203–206.

Let $f(z)=\alpha_0+\alpha_1z+\cdots$ be univalent in $|z|<1$ and let $M(r,f)=\max_{|z|=r}|f(z)|$. In the paper reviewed above the reviewer has shown that $(1-r)^2M(r,f)\to\alpha\leqq 1$ as $r\to 1$. The author now shows that $(1-r)^3M(r,f')\to 2\alpha$ as $r\to 1$.
W. K. Hayman (Exeter).

Jenkins, James A. On circumferentially mean p-valent functions. Trans. Amer. Math. Soc. **79** (1955), 423–428.

If p is a positive integer, then F_p denotes the class of all power series $w=f(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+\cdots$ in $|z|<1$ that are circumferentially mean p-valent; i.e., the total length, on each $|w|=r$, covered in the Riemann map of $|z|<1$ by $f(z)$ is at most $2\pi p$. Several sharp estimates are obtained, amongst them $|a_{p+2}|\leqq 2p^2+p$ with equality for $z^p(1-ze^{i\theta})^{-2p}$. [$|a_{p+1}|\leqq 2p$ was proved by Spencer, Ann. of Math. (2) **42** (1941), 614–633; MR **3**, 78]. The author uses, as in his previous work [Amer. J. Math. **75** (1953), 510–522; MR **15**, 115], the method of symmetrization. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Unkelbach, Helmut. Geometrie und konforme Abbildung verallgemeinerter Kreisbogenpolygone. I. Math. Ann. **129** (1955), 391–414.

Continuing his work [cf. Math. Ann. **125** (1952), 82–118; MR **14**, 462] on generalization of conformal mapping of polygonal domains, the author extends the concept of a circular polygon by admitting any simply-connected domain on a Riemann surface which contains no branch-points in its interior and which is bounded by a finite number of circular arcs of any sheetedness. It is shown that, for a function $z=f(\zeta)$ mapping the upper half of the ζ-plane onto such a polygon, the classical Schwarz differential equation $\{z,\zeta\}=F(\zeta)$ remains valid, where $\{z,\zeta\}$ denotes the Schwarzian derivative and $F(\zeta)$ is a rational function with real coefficients which possesses no poles except at the points corresponding to vertices of the polygon. Properties of some coefficients of $F(\zeta)$ are investigated. *Y. Komatu* (Tokyo).

Akaza, Tôru. On the subregions bounded by the level curves of the Green's function. Sci. Rep. Kanazawa Univ. 3 (1955), no. 1, 1–3.

$w'=T(w)$ désignant une function holomorphe de w, on dit qu'un domaine R possède la propriété T si $w\in R\Rightarrow T(w)\in R$. Si T est univalente dans R, avec $T(0)=0$, $|T'(0)|\leqq 1$, et si R a la propriété T, il en est de même pour les domaines R_γ définis par $G(w,0)>-\log\gamma$ (G étant la fonction de Green de R; $0<\gamma<1$). Ce résultat étend une remarque de J. L. Walsh [Duke Math. J. **20** (1953), 611–615; MR **15**, 310]. L'A. étudie ensuite le groupe G des transformations biuniformes de R sur lui-même, laissant un point invariant, au moyen d'une représentation conforme de R^∞ sur $|z|<1$; mais il n'est pas évident, ainsi que l'affirme l'auteur, que le groupe G soit toujours cyclique, et ceci est certainement inexact si le point fixe est un point frontière, le plan pointé par les points $p/(p+1)$ et $(p+1)/p$ (p entier) fournissant un contre exemple. *J. Lelong* (Lille).

Hiong, King-Lai. Sur un théorème fondamental de M. Milloux. C. R. Acad. Sci. Paris **241** (1955), 271–273.

The author establishes under appropriate conditions an inequality of the type

$$T(r,f)<N(r,f)+N\left(r,\frac{1}{f-\phi}\right)+N\left(r,\frac{1}{f_l-\psi}\right)-N_1(r,f)+S(r)$$

in the usual notation of the Nevanlinna theory. Here $f_l=\sum_{i=0}^{l}\alpha_if^{(i)}$, and ϕ, ψ and the α_i are meromorphic functions. *W. K. Hayman* (Exeter).

Noshiro, Kiyoshi. Cluster sets of functions meromorphic in the unit circle. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 398–401.

Let $w=f(z)$ be a meromorphic function in $D: |z|<1$, let $z_0=e^{i\theta_0}$ be a fixed point on $C: |z|=1$, A an open arc on C containing z_0, E a set of measure zero containing z_0 and contained in A. With every $e^{i\theta}\in A-E$ there is associated an arbitrary simple curve $\Lambda(e^{i\theta})$ in D terminating in $z=e^{i\theta}$ and the cluster set $S_\Lambda(e^{i\theta})$ of $f(z)$ at $z=e^{i\theta}$ with respect to $\Lambda(e^{i\theta})$. A boundary cluster set $\Gamma_{z_0}^{(C)}$ of $f(z)$ at z_0 is defined by $\Gamma_{z_0}^{(C)}=\bigcap_{r>0}M_r$, where $M_r=\overline{\bigcup S_\Lambda(e^{i\theta})}$ and the union is extended over all $e^{i\theta}\in(A-E)\cap\{|z|<1\}$. The relations of $\Gamma_{z_0}^{(C)}$ to $S_{z_0}^{(D)}$, the interior cluster set of $f(z)$ at z_0, and to $R_{z_0}^{(C)}$, the range of values of $f(z)$ at z_0, are investigated [for the definition of these terms, see Collingwood and Cartwright, Acta Math. **87** (1952), 83–146, pp. 120–121; MR **14**, 260]. It is proved that $\Omega_0=S_{z_0}^{(D)}-\Gamma_{z_0}^{(C)}$ is an open set. Further results announced are: 1. If $\alpha\in\Omega_0$ is an exceptional value of $f(z)$ in a neighborhood of z_0, then either α is an asymptotic value of $f(z)$ at z_0, or there exists a sequence of points $z_n\in C$, with $z_n\to z_0$, such that α is an asymptotic value of $f(z)$ at each z_n. 2. If Ω_0 is not empty, then $R_{z_0}^{(D)}$ covers Ω_0, except possibly for a set of capacity zero. 3. Let H be a given closed set in the w-plane, Ω_1 a component of its comple-

ment, and α an accessible boundary point of Ω_1. For every $e^{i\theta} \epsilon A-E$, let $S_\Delta(e^{i\theta})$ be contained in H and let $S_{z_0}^{(D)} \cap \Omega_1$ be not empty. Furthermore, let the set of all $e^{i\theta} \epsilon A-E$ with $\alpha \epsilon S_\Delta(e^{i\theta})$ be of measure zero. Then if α is an exceptional value of $f(z)$ in some neighborhood of z_0, α is an asymptotic value of $f(z)$ arbitrarily near z_0. These theorems generalize recent results of Ohtsuka [J. Math. Soc. Japan **2** (1950), 1–15; MR **13**, 336], Lohwater [Duke Math. J. **19** (1952), 243–252; C. R. Acad. Sci. Paris **237** (1953), 16–18; MR **14**, 34; **15**, 517], Lehto [Ann. Acad. Sci. Fenn. Ser. A. I. no. **160** (1953); MR **15**, 517], and Storvick [Bull. Amer. Math. Soc. **60** (1954), 481].

W. Seidel (Notre Dame, Ind.).

Gol'dberg, A. A. On defects of meromorphic functions. Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 893–895. (Russian)

The author answers in affirmative one of the greater open questions within the theory of distribution of values: He constructs a class of meromorphic functions, with Nevanlinna-defects $\delta(a_\nu)>0$ for a (given) denumerable set a_ν. His method of condensing exponentials yields a function of finite order (order 1); it is similar to a process, which enabled Valiron about thirty years ago to construct functions with "many" Valiron-defects $\Delta(a)$. The functions, showing the quality desired, are all meromorphic in the stronger sense, the corresponding question for entire functions remaining open. Furthermore: A meromorphic canonical product of genus zero, such as

$$f(z)=\prod(1-za_n^{-1}):\prod(1-zb_\lambda^{-1})$$

shows defects not greater than those of the corresponding function $\bar{f}(z)$ with zeros $A_n=|a_n|$ and poles $B_\lambda=-|b_\lambda|$. This result allows interesting additions to questions, suggestions and theorems of Teichmüller [Deutsche Math. **4** (1939), 163–190].

E. Ullrich (Giessen).

Gol'dberg, A. A. On the influence of clustering of algebraic branch points of a Riemann surface on the order of growth of a meromorphic mapping function. Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 709–711; correction, **101** (1955), 4. (Russian)

The well known theorem of Denjoy-Carleman-Ahlfors (a meromorphic function whose Riemann surface contains n direct critical points is of order $n/2$, normal type, at least) is extended to a new class of critical points, called K-points, generalizing some more special results of Tsuji [Proc. Imp. Acad. Tokyo **18** (1942), 132–139; MR **8**, 203] and Tumura [ibid. **17** (1941), 65–69; **18** (1942) 606–611, 612–616; Jap. J. Math. **18** (1943) 797–876; MR **2**, 358; **7**, 288; **8**, 24]. K-points are defined in potential-theoretic terms, but in certain cases a geometrical interpretation is suggested.

E. Ullrich (Giessen).

Gheorghiu, Octavian Em. Extensions de la monogénéité au sens de V. S. Fédoroff. Com. Acad. R. P. Române **2** (1952), 673–676. (Romanian. Russian and French summaries)

If $f(z)$, $g(z)$ are complex functions of $z=x+iy$, then f is said to have a second-order derivative with respect to g provided that the expression

$$\frac{\Delta_2 f}{(\Delta g)^2}=\frac{f[x+2\Delta x)+i(y+2\Delta y)]-2f[(x+\Delta x)+i(y+\Delta y)]+f(x, y)}{\{g[(x+\Delta x)+i(y+\Delta y)]-g(x, y)\}^2}$$

has a limit as $\Delta z\to 0$; it is shown that a necessary and sufficient condition for this is that $\partial f/\partial x$ and $\partial f/\partial y$ be monogenic functions of g in the sense of Fedorov [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **48** (1945), 389–390; MR **8**, 25]. The derivative of order n, as limit of $\Delta_n f/(\Delta g)^n$, also is discussed, as are the areolar derivative of f with respect to g, as limit of $\int_C f(z)dg(z)/\iint_D dx\,dy$; mixed derivatives, as limits of expressions $\Delta_{zz'}f(z, z')/[\Delta g(z)\Delta g'(z')]$; etc.

E. F. Beckenbach (Los Angeles, Calif.).

Ozaki, Shigeo, Ono, Isao, and Umezawa, Toshio. General minimum problems and representative domains. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1955), 1–7.

The authors consider analytic functions $W=W(Z)$ in vector space, where $W\equiv(w_1, \cdots, w_k)'$ and $Z\equiv(z_1, \cdots, z_k)'$ are k-column vectors, and the elements w_i of W are analytic functions of all the elements z_i of Z in a bounded domain B of C_k. The authors first determine the function $F(Z)$ which minimizes the integral

$$J(F)\equiv\int_B F^*(Z)F(Z)d\omega\equiv\|F\|^2$$

under the conditions $W(t)=0$, $[dW/dZ]_{Z=t}\equiv E_k$, where $F^*(Z)$ is the transformed and conjugate vector of $F(Z)$, $d\omega$ is the volume element in B, t is a given point in B, dW/dZ is the matrix

$$\begin{bmatrix} \frac{\partial w_1}{\partial z_1} & \cdots & \frac{\partial w_1}{\partial z_k} \\ \cdot & \cdots & \cdot \\ \frac{\partial w_k}{\partial z_1} & \cdots & \frac{\partial w_k}{\partial z_k} \end{bmatrix}$$

and E_k is the unit matrix of order k. They determine the minimizing function F in terms of a complete orthonormal system $\{\phi_\nu(Z)\}$ in $L^2(B)$. Having this minimizing function, the authors then obtain Bergman's representative domain for this class of vector functions.

W. T. Martin (Cambridge, Mass.).

On order

★Lammel, Ernesto. Generalizations of the theory of functions of a complex variable. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 191–197. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

It is noted that, for the Laplace equation

$$\partial^2U/\partial x^2+\partial^2U/\partial y^2=0$$

the class of homogeneous polynomial solutions of degree n is the class of functions $a_0u_n(x, y)+a_1v_n(x, y)$, where u_n and v_n are the real and imaginary parts, respectively, of the nth power of $z=x+iy$, and a_0 and a_1 are arbitrary constants; further, it is noted that the rule of composition of these functions is expressed by

$$(u_{k+1}, v_{k+1})=(u_ku_1-v_kv_1,\ u_kv_1+u_1v_k),$$

which is exactly the rule for the multiplication of complex numbers. The author uses the foregoing observations as a basis for setting up partial analogues of the theory of analytic functions of a complex variable from the rules of composition of homogeneous polynomial solutions of other partial differential equations. Thus to the equation $\partial^2U/\partial x^2-\partial^2U/\partial y^2=0$ there corresponds the rule

$$(a_1, b_1)(a_2, b_2)=(a_1a_2+b_1b_2,\ a_1b_2+a_2b_1).$$

E. F. Beckenbach (Los Angeles, Calif.).

Chang, Hsiao-li. Approximately analytic functions of bounded type and boundary behaviour of solutions of elliptic partial differential equations. Acta Math. Sinica 3 (1953), 101–132. (Chinese. English summary)

A complex-valued function w of a complex variable $z=x+iy$ is called approximately analytic if it satisfies the differential inequality $|w_x+iw_y|\leq\text{const.}\times|w|$. Such functions have many properties in common with analytic functions [cf. Bers, Theory of pseudoanalytic functions, Inst. Math. Mech., New York Univ., 1953; MR 15, 211]. In this paper the author defines approximately analytic functions "of bounded type" which turn out to be quotients of bounded approximately analytic functions. He derives a generalization of the Poisson-Jensen formula and proves analogues of some classical theorems in function theory. These theorems are then applied to the study of boundary behavior of functions satisfying a linear partial differential equation of elliptic type and of second order. For such functions the author proves results which generalize the theorems due to Fatou, F. and M. Riesz, and Blaschke. *L. Bers* (New York, N.Y.).

Roşculeţ, Marcel N. Algèbres linéaires associatives et commutatives et fonctions monogènes qui y sont attachées. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 6 (1955), 135–173. (Romanian. Russian and French summaries)

This paper is concerned with linear, associative, commutative algebras of order $n+1$ over the real or complex field with basis $\theta_0, \theta_1, \cdots, \theta_n$ such that θ_0 is the identity. Let $\theta_i\theta_j=\sum_{k=0}^{n} a_{ijk}\theta_k$. Using the commutative and associative laws, the author shows that all the a_{ijk} may be evaluated in terms of $n(n+1)$ constants a_{1jk} ($j=1, 2, \cdots, n$; $k=0, 1, \cdots, n$). In terms of these parameters the author determines the characteristic equation of θ_1 and the structure of the algebra.

For $\omega=\sum x_i\theta_i$ the author determines expressions for $\exp\omega$ and $\cos\omega$. Convergence of power series is also discussed.

A function $f(\omega)$ is defined to be monogenic in a region D if $\theta_k(\partial f(\omega)/\partial x_i)=\theta_i(\partial f(\omega)/\partial x_k)$ for all i and k. A monogenic function of a monogenic function is proved to be monogenic. Generalizations of contour integration and the Laplace differential equation are developed. The theory is also extended to functions of several hypercomplex variables. *J. A. Ward* (Holloman, N.M.).

See also: Makar, p. 149; Makar and Hamza, p. 149; Moisil, p. 155.

Harmonic Functions, Potential Theory

Berstein, I. Une caractérisation topologique de la pseudo-conjuguée d'une fonction pseudo-harmonique. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 75–78. (Romanian, Russian, and French summaries)

A real-valued function $v(p)$ on a surface S is called a pseudo-conjugate of the pseudo-harmonic function u if $u+iv$ is an interior mapping of S to the complex plane. The present paper establishes the result that v is pseudo-conjugate to u if and only if v is an interior mapping into the real line on each continuum where $u(p)$ is constant. *H. L. Royden* (Stanford, Calif.).

Ishikawa, Osamu. On the characterization of the harmonic functions. Proc. Japan Acad. 30 (1954), 686–690.

The author establishes the following result: If $u(x, y)$ is of class C' in a domain R, $v(x, y)$ is harmonic and $\neq 0$ in R, and the relation

$$(*)\quad 2\int_0^{2\pi}\int_0^r\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\varrho d\varrho d\theta -\int_0^{2\pi}\left(v\frac{\partial u}{\partial n}+u\frac{\partial v}{\partial n}\right)r d\theta=o(r^2)$$

holds throughout R, then $u(x, y)$ is harmonic in R. The reviewer notes that (*) is the special case $\lambda=1$ of the relation

$$(1+\lambda)\int_0^{2\pi}\int_0^r\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\varrho d\varrho d\theta -\int_0^{2\pi}\left(v\frac{\partial u}{\partial n}+\lambda u\frac{\partial v}{\partial n}\right)r\, d\theta=o(r^2),$$

which characterizes harmonic functions $u(x, y)$ for any constant λ; the case $\lambda=-1$ has been treated by Gergen [Bull. Amer. Math. Soc. 37 (1931), 591–596] and Saks [ibid. 38 (1932), 380–382]. Using a lemma due to the reviewer [Proc. Amer. Math. Soc. 3 (1952), 765–769; MR 14, 272], the author shows that the same conclusion follows if the foregoing condition $v\neq 0$ is replaced by the conditions that $u=0$ whenever $v=0$, and that $\partial^2u/\partial x^2$ and $\partial^2u/\partial y^2$ are summable.

The following result also is established: If $u(x, y)$ satisfies the relation

$$(**)\quad \frac{1}{2\pi}\int_0^{2\pi}u(x+r\cos\theta, y+r\sin\theta)d\theta -\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r u(x+\varrho\cos\theta, y+\varrho\sin\theta)\varrho\, d\varrho d\theta=o(r^2)$$

throughout R, then $u(x, y)$ is harmonic in R provided that $u(x, y)$ is of class C' and the rates of change of $\partial u/\partial x$ and $\partial u/\partial y$ satisfy a certain boundedness condition. It is conjectured that if $u(x, y)$ is assumed only to be of class C and to satisfy (**) throughout R then $u(x, y)$ must be harmonic in R. The conjectured result actually is valid under an even less restrictive hypothesis; see the following review and also papers by Privalov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 31 (1941), 102–103; MR 2, 366], Reade [Duke Math. J. 10 (1943), 531–536; MR 5, 7], and Aki [Kōdai Math. Sem. Rep. 1950, 11–12; MR 12, 258]; the reviewer observes that the unproved statement noted in the review of the last of these papers actually is false. *E. F. Beckenbach* (Los Angeles, Calif.).

Tashiro, Shizuko. On almost subharmonic functions. Mem. Fac. Sci. Kyûsyû Univ. Ser. A. 9 (1955), 55–59.

A function f is almost subharmonic in a simply connected domain D in euclidean n-space provided that f is defined almost everywhere in D, is locally summable, and almost everywhere in D satisfies the inequality $f\leq\mu_\varrho * f$ for all $\varrho>0$ such that the sphere S: $|x|=\varrho$ is in D, where μ_ϱ is the measure due to a unit mass distributed homogeneously on S [see Schwartz, Théorie des distributions, t. II, Hermann, Paris, 1951, p. 76; MR 12, 833]. It is now shown that, in the class of functions defined almost everywhere in D and locally summable, almost subharmonic functions are characterized by the inequality $(\nu_\varrho-\delta)*f\geq 0$ for all spheres in D, where ν_ϱ is the measure due to a unit mass distributed homogeneously on $|x|\leq\varrho$.

and δ is the Dirac measure. Similarly, almost subharmonic functions are shown to be characterized, in the same class of functions, by the inequality $(\mu_\varrho-\nu_\varrho)*f\geqq 0$; this latter theorem generalizes a result of the reviewer [Univ. Nac. Tucumán. Rev. Ser. A. 4 (1944), 243–254; MR 7, 122], who established it only for $n=2$, and under the assumption that f is superficially summable.

E. F. Beckenbach (Los Angeles, Calif.).

Hayashi, Kenji. On the function almost of class PL. Mem. Fac. Sci. Kyûsyû Univ. Ser. A. 9 (1955), 61–64.

A function u is almost of class PL provided that u is $\geqq 0$ and summable, and provided that $\log u$ is almost subharmonic [see the preceding review]. The author establishes the result that, in the class of functions which are $\geqq 0$, are locally summable, and have locally summable logarithms, the functions u which are almost of class PL are characterized by the property that, for all α $(0<\alpha\leqq 1)$, the function u^α is almost subharmonic [see T. Radó, Subharmonic functions, Springer, Berlin, 1937, for the corresponding result concerning functions of class PL].

E. F. Beckenbach (Los Angeles, Calif.).

Targonszky, G., und Bognár, Z. Über die Bestimmung konjugierter harmonischer Funktionen. Publ. Math. Debrecen 3 (1954), 215–216 (1955).

If the analytic function $f=u+iv$ is real on the real axis, the formal identity $u(x, y)+iv(x, y)=u(x+iy, 0)$ makes it easy to compute v from u.

W. Rudin (Rochester, N.Y.).

Lavrent'ev, M. M. On Cauchy's problem for Laplace's equation. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 205–206. (Russian)

Let the function $u(x, y)$ be harmonic in the strip $0\leqq y\leqq 1$ and satisfy the following condition: $u(x, 0)=0$; $u(x+2k\pi, y)=u(x, y)$; $u(-x, y)=u(x, y)$; $u(k\pi, y)=0$; $|u(x, y)|<M$; and $\partial u(x, 0)/\partial y=\varphi(x)$. Further, let the sequence of functions $w_n(x, y)$, defined in the same strip, satisfy the first four of the foregoing conditions and also $\partial w_n(x, 0)/\partial y=\varphi_n(x)$, where $\varphi_n(x)$ converge uniformly to $\varphi(x)$ as $n\to\infty$. It is shown that if $w_n(x, y)$ satisfies the partial differential equation

$$-\varepsilon_n\frac{\partial^{2m}w_n}{\partial x^{2m}}=\frac{\partial^2 w_n}{\partial x^2}+\frac{\partial^2 w_n}{\partial y^2},$$

then, for some values of the ε_n, the sequence of functions $w_n(x, y)$ must converge to the harmonic function $u(x, y)$. An analogous reasoning shows that if the ε_n all vanish, that is, if the $w_n(x, y)$ are harmonic, then the sequence converges to $u(x, y)$ provided that we omit the hypothesis that the functions vanish on the x axis.

E. F. Beckenbach (Los Angeles, Calif.).

Di Palo, Raffaele. Sul problema di Dirichlet in un campo prossimo ad une sfera. Ricerca, Napoli 5 (1954), no. 4, 27–32.

Verf. betrachtet das zur Laplace'schen Gleichung gehörige Dirichletsche Problem für eine der Kugel σ benachbarte Randfläche σ', für deren Punkte Q' eine umkehrbar eindeutige Zuordnung zu den Punkten Q von σ durch eine Beziehung der Form $Q'=Q+\varepsilon(Q)\mathbf{n}$ ($\mathbf{n}$ Einheitsvektor in Richtung des Kugelradius) besteht; $\varepsilon(Q)$ ist hierbei eine auf σ gegebene Funktion. Die Lösung des Dirichletschen Problems für σ' wird für kleine $|\varepsilon(Q)|$ angenähert als Differenz zweier passender Poissonscher Integrale für σ dargestellt.

K. Maruhn (Dresden).

af Hällström, Gunnar. On the capacity of generalized Cantor sets. Acta Acad. Abo. 20 (1955), no. 5, 8 pp.

The generalized Cantor sets E of the interval [0, 1] are defined by prescribing the sequences $\{k_n\}$, $\{p_n\}$ where $k_n\geqq 2$ are integers, $p_n>1$. Let $N_n=k_1k_2\cdots k_n$, $P_n=p_1p_2\cdots p_n$, $d_n=(N_nP_n)^{-1}$. We denote by $E_n=\{e_{nm}\}$, $m=1, 2, \cdots, N_n$, the set of closed intervals e_{nm} of equal lengths, d_n arising in the following inductive manner. We remove from $e_{n-1,\nu}$ in the whole k_n-1 open gaps, each having the same length δ_n. (It is easy to express δ_n in terms of k_ν and p_ν.) Now $E=\bigcap E_n$. The author proves the following theorems. If $\sum N_n^{-1}\log p_n=\infty$, the set E is of capacity 0. If both the previous series and the series $\sum N_{n-1}^{-1}\log k_n$ are convergent, E has a positive capacity.

G. Szegö.

See also: Položil, p. 211.

Series, Summability

Pitt, H. R. A note on Tauberian conditions for Abel and Cesàro summability. Proc. Amer. Math. Soc. 6 (1955), 616–619.

In answer to a question of Butzer a non-trivial condition (τ) on $s(x)$, $x>0$, is found which is a Tauberian condition for $(C, 1)$, but not a Tauberian condition for the Abel method of summation. The condition (τ) is as follows. For each $\varepsilon>0$ there is an $\eta>0$ such that for all sufficiently large x there are y, R satisfying $R\geqq 1+\eta$, $y\leqq x\leqq Ry$ such that

$$\left|\int_y^{Ry}[s(u)-s(x)]du\right|\leqq(R-1)y\varepsilon.$$

G. G. Lorentz (Detroit, Mich.).

Agnew, Ralph Palmer. Permutations preserving convergence of series. Proc. Amer. Math. Soc. 6 (1955), 563–564.

In answer to question raised by R. C. Buck, the author proves the following theorem. For a permutation $(p_i)_1^\infty$ of the sequence of (positive) integers, the following two statements are equivalent: (1) wherever $\sum a_k$ is a convergent series of real numbers, $\sum_{k=1}^\infty a_{p_k}=\sum_{k=1}^\infty a_k$; (2) there is an integer N such that for each n, the set $(p_i)_1^n$ is the union of N or fewer sets of consecutive integers.

V. L. Klee (Los Angeles, Calif.).

Agnew, Ralph Palmer. Equiconvergence of Cesàro and Riesz transforms of series. Duke Math. J. 22 (1955), 451–460.

Given a series (1) $\sum u_n$ with partial sums s_n, let

$$R_r(\omega)=\sum_{k<\omega}\left(1-\frac{k}{\omega}\right)^r u_k\ (\omega>0)\text{ and}$$

$$C_r(n)=\sum_{k=0}^n\left[\binom{n-k+r}{n-k}\bigg/\binom{n+r}{r}\right]u_k$$

be the Riesz and Cesàro means of (1) for fixed $r>0$. It is shown that (2) $\lim_{n\to\infty}[C_r(n)-R_r(n)]=0$ and (3) $\lim_{\omega\to\infty}[C_r([\omega])-R_r(\omega)]=0$ provided the terms of (1) satisfy $\lim u_n=0$ or $\limsup|x_n|<\infty$ $[x_n=(n+1)^{-1}\sum_{k=0}^n ku_k]$. To prove (3) from (2) it is established that

$$\lim[R_r(\omega_1)-R_r(\omega_2)]=0$$

for all series (1) with $\limsup|x_n|<\infty$ if $\omega_1\to\infty$, $\omega_2\to\infty$, $\omega_1/\omega_2\to 1$, and that $\lim[R_r(\omega_1)-R_r(\omega_2)]=0$ for all series (1) with $\lim u_n=0$ if and only if $\omega_1\to\infty$, $\omega_2\to\infty$,

lim sup $|\omega_1-\omega_2|<\infty$. Theorems (2) and (3) show, in particular, that results of Garten [Comment. Math. Helv. **25** (1951), 311–335; MR **13**, 548] on Tauberian constants relating $\{s_n\}$ and its Cesàro transform are equivalent to those of the author [e.g., J. Reine Angew. Math. **193** (1954), 94–118; MR **16**, 237] on Tauberian constants relating $\{s_n\}$ and its Riesz transform. *D. Gaier* (Stuttgart).

Lorentz, G. G. Borel and Banach properties of methods of summation. Duke Math. J. 22 (1955), 129–141.

A regular method of summability $A=(a_{mn})$ has the Borel property ($A \in$ BP) if (i) $\lim_m \sum_n a_{mn}r_n(x)=0$ almost everywhere (a.e.) where $r_n(x)$ is the nth Rademacher function; A has the Banach property if $\lim_m \sum_n a_{mn}\phi_n(x)=0$ a.e. for every normal orthogonal system $\phi_n(x)$ on (0, 1). If (i) holds for all sequences $r_n(x)$ of uniformly bounded independent functions with $\int_0^1 r_n(x)dx=0$, one says that $A \in$ BP*. The author connects these concepts with his earlier notion of a summability function and derives sufficient conditions for the Banach and Borel properties. A new necessary condition for the Borel property is given. Turning his attention to special methods the author establishes, for instance, the following theorems: (ii) If $\Delta\lambda_n$ increases and $\Delta\lambda_n/\lambda_n$ decreases to 0, a necessary and sufficient condition for the Riesz method $R(\lambda_n, k>0)$ or the Abel method $A(\lambda_n)$ to have the Borel property is that $\Delta\lambda_n \log n=o(\lambda_n)$. (iii) A regular Hausdorff method with generating function $g(x)$ will have the Borel property if and only if $g(x)$ is continuous at $x=1$. The reader will find the author's list of references complete and valuable.

J. D. Hill (East Lansing, Mich.).

Ganelius, Tord. On the remainder in a Tauberian theorem. Kungl. Fysiog. Sällsk. i Lund Förh. 24 (1954), no. 20, 6 pp. (1955).

Let $\alpha(\lambda)$ be a real function of bounded variation defined for $\lambda\geqq\lambda_0$ such that (i) $\alpha(\lambda_0)=0$, (ii) $\alpha(\lambda)+K(\lambda-\lambda_0)^\nu$ is ultimately non-decreasing, and (iii) $\int_{\lambda_0}^\infty e^{-\lambda t}d\alpha(\lambda)=O[\exp(-ct^{-\varepsilon})]$ for $t\to0+$, where $0<\varepsilon\leqq1$, K, c, and ν are positive constants and $\nu\geqq2\varepsilon/(1+\varepsilon)$. Then, if $\gamma\geqq1$,

$$\int_{\lambda_0}^x\left(\frac{x-\lambda}{x-\lambda_0}\right)^{\gamma-1}d\alpha(\lambda)=O(x^{\nu-\gamma\varepsilon/(1+\varepsilon)}) \text{ as } x\to\infty.$$

This theorem is proved with the aid of methods due to G. Freud [see Acta Math. Acad. Sci. Hungar. **2** (1951), 299–308; **3** (1953), 299–307; MR **14**, 361, 958]. An important application of the theorem is given in a paper by L. Gårding [same Förh. **24** (1954), no. 21; MR **17**, 158].

E. Hille (New Haven, Conn.).

Knopp, K., and Vanderburg, B. Functional Nörlund methods I. Rend. Circ. Mat. Palermo (2) 4 (1955), 5–32.

The authors consider Nörlund methods of summability applied to integrals and functions. A weight function is a real-valued measurable function $p(t)$ defined for $t>0$ and bounded in any interval (a, b), $0<a<b<\infty$, such that $P(x)=\int_0^x p(t)dt$ is absolutely convergent and >0 for all $x>0$. If $s(t)$ (possibly complex-valued) is bounded and measurable in $(0, c)$, $c>0$, then

$$\sigma(x)=[P(x)]^{-1}\int_0^x p(x-t)s(t)dt$$

is called the N_p-mean of $s(t)$. If $\sigma(x)\to s$ as $x\to\infty$, then $s(t)$ is said to be N_p-limitable and this is written $s(t)\to s$ (N_p) as $t\to\infty$. If $a(u)$ is given and $\int_0^t a(u)du=s(t)$, where $s(t)\to s$ (N_p), then the integral is said to be N_p-summable to s.

A variety of theorems are proved concerning the inclusion relations of two such methods, their connection with Abel summability and with the method obtained by using the resultant (Faltung) of two weight functions as a weight function. *H. G. Eggleston* (Cambridge, England).

See also: San Juan Llosá, p. 139; Henstock, p. 177.

Interpolation, Approximation, Orthogonal Functions

Hsu, L. C. Note on Maréchal's integral approximation. Acta Math. Sinica 3 (1953), 148–153. (Chinese. English summary)

From a theorem of J. E. Wilkins, Jr. [Bull. Amer. Math. Soc. **55** (1949), 191–192; MR **10**, 516], the formula

$$\lim_{t\to\infty}\int_a^b F(x, \cos tx, \sin tx)G(x)dx=\frac{1}{2\pi}\int_0^{2\pi}\int_a^b F(x, \cos\theta, \sin\theta)G(x)dx\,d\theta$$

is derived, where $G \in L(a, b)$ and $F(x, y, z)$ is continuous on $a\leqq x\leqq b$, $|y|\leqq1$, $|z|\leqq1$. Another similar result: Let D be the region bounded by continuous curves $y=f_1(x)$, $y=f_2(x)$ (where $f_1(x)\geqq f_2(x)$) and lines $x=a$, $x=b$. If F is continuous on D, then

$$\lim_{t\to\infty}\frac{\pi}{2}\int_a^b F(x, g_t(x))h_t(x)dx=\iint_D F(x, y)dxdy,$$

where $g_t(x)=\cos^2(tx/2)\,f_1(x)+\sin^2(tx/2)\,f_2(x)$ and $h_t(x)=|\sin tx|\cdot(f_1(x)-f_2(x))$. *Ky Fan* (Notre Dame, Ind.).

Angheluţă, Th. Applications de la différence divisée, mises sous la forme d'une intégrale. Com. Acad. R. P. Române 2 (1952), 117–124. (Romanian. Russian and French summaries)

Let $f(z)$ be a function of the complex variable z, regular in Ω, the smallest convex domain of the $n+1$ complex numbers $z_1, z_2, \cdots, z_{n+1}$. If $I=[z_1, \cdots, z_{n+1}, f(z)]$ stands for the divided difference of $f(z)$ over $z_1, \cdots, z_{n+1}$, Hermite proved that

$$(*)\qquad I=\int_0^1 dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}f^{(n)}(u)du,$$

where $u=(z_{n+1}-z_n)t_n+\cdots+(z_2-z_1)t_1+z_1$. The author gives a short proof of (*) and also proves that

$$(**)\qquad I=\int_0^1\cdots\int_0^1 f^{(n)}(u)\cdot u_1^{n-1}u_2^{n-2}\cdots u_{n-1}du_1\cdots du_n,$$

where

$$u=(z_{n+1}-z_n)u_1u_2\cdots u_n+(z_n-z_{n-1})u_1u_2\cdots u_{n-1}+\cdots+(z_2-z_1)u_1+z_1.$$

Hence follows easily the Darboux- Jensen-Montel formula [Montel, Ann. Scuola Norm. Sup. Pisa (2) **1** (1932), 371–384] $I=\lambda f^{(n)}(\xi)/n!$, $|\lambda|\leqq1$, ξ in Ω and related formulae. Using (**), the remainder in Lagrange's interpolation formula is obtained as

$$R(x)=f(x)-L_{n-1}(x)=\varphi(x)\int_0^1\cdots\int_0^1 f^{(n)}(v)v_1^{n-1}\cdots v_{n-1}dv_1\cdots dv_n$$

where $\varphi(x)=\prod_{i=1}^n(x-z_i)$ and

$$v=(x-z_n)v_1\cdots v_n+\cdots+(z_2-z_1)v_1+z_1$$

and an application is made to obtain the well-known error term of Simpson's integration scheme. Finally, an

expression of the divided difference [see Mikeladze, Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 3–88; MR 10, 575] over $m+n$ points, $z_1=z_2=\cdots=z_m$, $z_j\neq z_i$ for $m<j\neq i$ is established. *E. Grosswald.*

Kneschke, A. Über den Zusammenhang differentieller Integrationsprobleme mit den Interpolationstheoremen. Jenaer Jahrbuch 1952, 79–102.

Da die Interpolationstheoreme Lösungsformen von in bestimmter Art gestellten Integrationsaufgaben sind — z. B. das Lagrangesche Interpolationstheorem Lösungsform der Lagrangeschen Integrationsaufgabe, das Taylorsche Interpolationstheorem Lösungsform der Cauchyschen Integrationsaufgabe —, stellt sich Verf. die Aufgabe, für eine allgemeine lineare Differentialgleichung n-ter Ordnung den Interpolationstheoremen entsprechende Lösungsformen aufzustellen. Er behandelt zunächst die Cauchysche Integrationsaufgabe

$$P\{f\}\equiv a_0f+a_1f'+\cdots+a_nf^{(n)}=g(x)$$

mit $f=f(x)$, wo für die Integralkurve $f(x)$ an der Stelle x_0 die Werte $f(x_0)$, $f'(x_0)$, $\cdots$, $f^{(n-1)}(x_0)$ vorgegeben sind. Ihre Lösung läßt sich — wie Verf. zeigt — in der Form

$$f(x)=\sum_{\nu=0}^{n-1}f^{(\nu)}(x_0)\Delta^{\nu+1}\mathfrak{P}_\xi\{K(x,\xi)\}|_{\xi=x_0}+\int_{x_0}^{x}K(x,\xi)g(\xi)d\xi$$

darstellen, worin die Greensche Funktion $K(x,\xi)$ bestimmten näher angegebenen Bedingungen genügt, ferner $\mathfrak{P}_\xi\{\ \}=(\partial/\partial\xi)(\mathfrak{P}\{\ \})$ ist und $\mathfrak{P}\{f\}$ den zu $P\{f\}$ adjungierten Differentialausdruck $a_0f-(a_1f)'+\cdots(-1)^n(a_nf)^{(n)}$ bedeutet. Außerdem ist $\Delta^\nu\mathfrak{P}\{h\}=\sum_{\mu=\nu}^{n}(-1)^{\mu-\nu}[a_\mu h]^{\mu-\nu}$. Weiter zeigt Verf., wie sich $f(x)$ auch als Lösung der Volterraschen Integralgleichung 2. Art oder als Taylorsche Reihe mit Bernoullischem Integralrestglied bestimmen läßt. In entsprechender Weise untersucht Verf. die Lagrangesche Integrationsaufgabe, die Funktion $f(\xi)$ zu bestimmen, die der Differentialgleichung $P\{f(\xi)\}=g(\xi)$ genügt und an den Stellen x_1, x_2, $\cdots$, x_n die Werte $f(x_1)$, $f(x_2)$, $\cdots$, $f(x_n)$ annimmt, wo $\alpha=x_1<x_2<\cdots<x_n=\beta$ ist. Dabei ist $g(\xi)$ eine stetige freie Funktion. Als dritte „allgemeine Integrationsaufgabe" wird die Problemstellung dahin verallgemeinert, daß für die gesuchte Integrationskurve die Werte der Funktion und (bezw. oder) die Werte von Ableitungen der Funktion an bestimmten Stellen $x_1<x_2<\cdots<x_l$ vorgegeben sind, also n Werte der Art $f^{(\nu)}(x_\mu)$ mit $\mu=1$, $\cdots$, $l\leq n$ und $0\leq\nu\leq n-1$. Verf. erläutert die Ergebnisse der Untersuchung an einigen Beispielen. *J. Picht* (Zbl 52, 291).

Erdös, P., and Turán, P. On the role of the Lebesgue functions in the theory of the Lagrange interpolation. Acta Math. Acad. Sci. Hungar. 6, 47–66 (1955). (Russian summary)

Let $A=(x_{\nu n})$ $(\nu=1, 2, \cdots, n; n=1, 2, \cdots)$ be a triangular matrix of interpolation points where $-1\leq x_{\nu n}\leq 1$ and all $x_{\nu n}$ in one row are distinct. Let $f(x)$ be continuous; we form the polynomials

$$L_n(f)=\sum_{\nu=1}^{n}f(x_{\nu n})l_{\nu n}(x),$$

where $l_{\nu n}(x)$ denote the fundamental polynomials of the Lagrange interpolation. The authors investigate the following important class $A(\beta)$ of matrices A. There exists a number β, $0<\beta<1$, such that for the "Lebesgue constants" $M_n=\max_{-1\leq x\leq 1}|l_{\nu n}(x)|$ the following inequalities hold:

$$\limsup_{n\to\infty} M_n n^{-\beta-\epsilon}<c_1(\epsilon),\quad \limsup_{n\to\infty} M_n n^{-\beta+\epsilon}>c_2(\epsilon),$$

where $c_1(\epsilon)$, $c_2(\epsilon)$ are positive constants. The following results are obtained. (a) Let $\mu<\beta/(\beta+2)$; there exists an $f(x)\in\text{Lip}\,\gamma$ such that $L_n(f)$ is unbounded in $[-1, 1]$ as $n\to\infty$. (b) Let $\gamma>\beta$, $f(x)\in\text{Lip}\,\gamma$; then the sequence $L_n(f)$ is uniformly convergent in $[-1, 1]$. (c) Let $\gamma>\beta/(\beta+2)$; there exists a special matrix $A\in A(\beta)$ such that the corresponding $L_n(f)$ converge uniformly in $[-1, 1]$ whenever $f(x)\in\text{Lip}\,\gamma$. (d) Let $\gamma<\beta$; there exists a special matrix $A\in A(\beta)$ and a special $f(x)\in\text{Lip}\,\gamma$ such that $L_n(f)$ is unbounded in $[-1, 1]$. *G. Szegö* (Stanford, Calif.).

Mergelyan, S. N. Several remarks on S. N. Bernšteĭn's approximation problem. Akad. Nauk Armyan. SSR. Dokl. 20 (1955), 113–119. (Russian. Armenian summary)

Let $h(x)$ be a non-negative weight function $(-\infty<x<\infty)$. Put $M_h(z)=\sup|P(z)|$ where P runs through all polynomials satisfying $h(x)|P(x)|\leq 1$ $(-\infty<x<\infty)$. The author proved recently: If C_h is the Banach space of functions $g(x)$ continuous on the real axis and such that $h(x)g(x)\to 0$ as $|x|\to\infty$ with the norm $\|g\|=\sup_x h(x)|g(x)|$, then the polynomials in x are fundamental in C_h if and only if $M_{h(x)/(1+|x|)}(z)=\infty$ for all non-real z [Dokl. Akad. Nauk SSSR (N.S.) 97 (1954), 597–600; MR 16, 1104]. In this paper the author points out that this does not yet give a very explicit solution to the problem of approximability by weighted polynomials. He underlines this remark by proving the following odd facts: a) If $\mu(x)\geq 0$, $\lim_{|x|\to\infty}\mu(x)=0$, then there exists an $h(x)$ such that the polynomials are fundamental in $C_{\mu h}$, but not in C_h. b) There is an $h(x)$ such that $\{x^k\}_{k=0}^{\infty}$ is fundamental in C_h, but $\{x^k\}_{k=1}^{\infty}$ is not. c) Changing $h(x)$ at one point may change from fundamentality to the contrary. d) If $\lim_{|x|\to\infty}\nu(x)|x|^k=0$ $(k=0, 1, \cdots)$, then there exists an $h(x)$ such that $\{x^k\}_{k=0}^{\infty}$ is fundamental in C_h and $\limsup h(x)/\nu(x)>0$.

All the weight functions used in the constructions vanish except on a discrete set of points.

The paper concludes with some remarks about interpolation series

$$\sum_{n=1}^{\infty}\frac{C_nG(z)}{G'(\lambda_n)(z-\lambda_n)},$$

where the λ_n are real numbers satisfying $|\lambda_{n+1}/\lambda_n|\geq k>1$ and $G(z)=\prod_{k=1}^{\infty}(1-z/\lambda_k)$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Surányi, J., and Turán, P. Notes on interpolation. I. On some interpolatorical properties of the ultraspherical polynomials. Acta Math. Acad. Sci. Hungar. 6 (1955), 67–80. (Russian summary)

The authors are concerned with interpolation polynomials $f(x)$ arising from data which involve non-consecutive derivatives. In particular, the problem $f(x_\nu)=y_\nu$, $f''(x_\nu)=y_\nu''$ $(\nu=1, 2, \cdots, n)$ is considered. Assuming that n is odd and the x_ν are symmetrically located in $[-1, 1]$, there is in general no polynomial $f(x)$ of degree $2n-1$ satisfying the conditions. Moreover, if there is such a polynomial, there are infinitely many of them. In the further course of the paper the x_ν are chosen as the zeros of certain ultraspherical polynomials $P_n^{(\lambda)}(x)$. Let n be even and let the x_ν be the zeros of $(1-x^2)P'_{n-1}(x)$. Then the interpolation polynomial $f(x)$ of degree $2n-1$ is always unique. However, the following is true: Let x_ν be the zeros of $P_n^{(\lambda)}(x)$, $\lambda+\frac{1}{2}=$even integer >0, $n\geq\lambda+5/2$; there exist polynomials $f(x)\not\equiv 0$ of degree $n+\lambda+\frac{1}{2}$ such that $f(x_\nu)=f''(x_\nu)=0$. *G. Szegö* (Stanford, Calif.).

Brousse, Pierre. Sur la sommabilité (A) des séries ultrasphériques. C. R. Acad. Sci. Paris **241** (1955), 351–353.

Let $\mu > \frac{1}{2}$, and let $(\sin\theta)^{2\mu}g(\theta)$ be absolutely integrable in $[0, \pi]$. The Abel summability of the ultraspherical expansion of $g(\theta)$ in terms of $P_n^{(\mu)}(\cos\theta)$ is discussed. The results are generalizations of classical theorems on trigonometric series. The following application is stated. Let $W(x, y)$ be a regular solution of the differential equation $\Delta W + 2\mu y^{-1}W_y = 0$ in the semi-circle $x^2+y^2<1$, $y>0$, such that $y^{2\mu-1}W \to 0$ as $y \to 0$, $-1<x<1$. If the set on which $W(x, 0)=0$ has a point of accumulation in $-1<x<1$, then $W(x, y)$ must vanish identically. *G. Szegö.*

Endl, Kurt. Sur les systèmes de polynomes orthogonaux en involution. C. R. Acad. Sci. Paris **241** (1955), 682–684.

This paper deals with orthonormal sets of polynomials $P_n = \sum_{r=0}^n \pi_{nr}x_r$ obtained by orthogonalizing the sequence $\{x_n\}_0^\infty$, $x_n(t)=t^n$, with respect to measures m on the real line and gives a partial answer to the following question: For what measures can one choose a sequence $\{c_n\}_0^\infty$ of real numbers so that the orthogonal system $\{c_nP_n\}_0^\infty$ is involutive in the sense that if $R_n = c_nP_n = \sum_{r=0}^n \varrho_{nr}x_r$, then $x_n = \sum_{r=0}^n \varrho_{nr}R_r$? If $(u, v) = \int u(t)v(t)dm(t)$, then the necessary and sufficient condition for this to happen is that $c_nc_r\pi_{nr} = (x_n, P_r)$ for $0 \leqq r \leqq n = 0, 1, 2, \cdots$, or, equivalently, that $c_n\pi_{nn} = \varepsilon_n = \pm 1$ and $\varepsilon_n\varepsilon_r\pi_{nr} = (x_n, P_r)\pi_{nn}\pi_{rr}$ for $0 \leqq r \leqq n = 0, 1, 2, \cdots$. The author then points out that the choices $\varepsilon_n = 1$ and $\varepsilon_n = (-1)^{[n/2]}$ give rise to involutive systems in the cases of the normalized Laguerre and Hermite polynomials, respectively, while the Gegenbauer, Jacobi, Legendre, and Tchebycheff orthonormal polynomials do not generate involutive systems.

A. E. Livingston (Seattle, Wash.).

Endl, Kurt. Sur une classe de polynomes orthogonaux généralisant ceux de Laguerre et de Hermite. C. R. Acad. Sci. Paris **241** (1955), 723–724.

Let E be the set of the rays $\arg z = 2\pi v/k$ ($v=0, 1, \cdots, k-1$), and $\omega(z)$ a weight function which is invariant under the rotation by $2\pi/k$. Using the scalar product

$$(f(z), g(z)) = \int_E f(z)\overline{g(z)}\omega(z)|dz|$$

the author orthogonalizes the powers $\{z^n\}$. The resulting polynomials $\{p_n(z)\}$ contain only powers of the form z^n, z^{n-k}, z^{n-2k}, $\cdots$. The case $\omega(z) = \exp(-z^k)$ is discussed. *G. Szegö* (Stanford, Calif.).

Rogosinski, W. W. Some elementary inequalities for polynomials. Math. Gaz. **39** (1955), 7–12.

Given a set of real numbers x_i ($0 \leqq i \leqq n$), with $x_0 < x_1 < \cdots < x_n$, and a set of non-negative masses M_i attached to the x_i, the author considers polynomials p of degree not exceeding n such that $|p(x_i)| \leqq M_i$ ($0 \leqq i \leqq n$). Using very simple arguments based on the Lagrange interpolation formula, the author obtains some general results concerning these polynomials which contain as special cases certain classical inequalities involving Tchebychef polynomials. Theorem 1. Let P be the polynomial of degree at most n determined by $P(x_i) = (-1)^{n-i}\cdot M_i$ ($0 \leqq i \leqq n$); then for the polynomials p introduced above, $|p^{(k)}(x)| \leqq |P^{(k)}(x)|$ ($0 \leqq k \leqq n$), for x outside the open interval (x_0, x_n). Equality holds only when $p(x) = cP(x)$, $|c|=1$; provided, when $k=0$, that $x \neq x_0$ and $x \neq x_n$. A number of corollaries are given. In particular, if $P(x) = \sum_0^n A_kx^k$ and $p(x) = \sum_0^n a_kx^k$, then $|a_n| \leqq |A_n|$. Applying this to a "Tchebychef distribution", $x_i = \cos\{(n-i)\pi/n\}$, $M_i = 1$, $0 \leqq i \leqq n$, one gets Tchebychef's inequality, $|a_n| \leqq 2^{n-1}$, $n \geqq 1$. The author then considers mass-distributions that are symmetric in the origin. For such distributions he proves among other results that: $|a_k| \leqq |A_k|$ if $k \equiv n \pmod 2$. Applying this result to the Tchebychef distribution and denoting by A_k^n the coefficient of x^k in the nth Tchebychef polynomial, he obtains: $|a_k| \leqq |A_k^n|$ for $k \equiv n \pmod 2$, an inequality due to W. Markoff for real polynomials with $|p(x)| \leqq 1$ for $|x| \leqq 1$. *J. Wermer.*

Makar, Ragy H. Some remarks on product and inverse basic sets of polynomials. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17** (1955), 466–469.

The author gives a simplified derivation, and generalizations, of some recent results on properties of products and inverses of basic sets of polynomials. *R. P. Boas, Jr.*

Makar, Ragy H., and Hamza, A. On basic sets of polynomials with given coefficients. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17** (1955), 438–448.

The authors investigate basic sets of polynomials whose coefficient matrices are composed of blocks of upper triangular matrices along the principal diagonal, with every element of the principal diagonal unity (and every element below it zero). They prove numerous results on the expansion properties of such sets under various further hypotheses on the coefficients. *R. P. Boas, Jr.*

See also: Nikol'skiĭ, p. 175.

Trigonometric Series and Integrals

Stečkin, S. B. On absolute convergence of Fourier series. II. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 221–246. (Russian)

[Part I appeared in the same Izv. Ser. Mat. **17** (1953), 87–98; MR **14**, 974.] Let A denote the class of periodic functions with (everywhere) absolutely convergent Fourier series. Given a class M of periodic continuous functions, the author asks: (1) Does M contain functions not in A? (2) Does M contain functions f such that $\tilde{f}$ (the conjugate function) is also in M but f is not in A? (3) Does M contain an f such that the Fourier series of f and of $\tilde{f}$ have no point of absolute convergence? (4) Same as (3) with both f and $\tilde{f}$ in M. The author answers all these questions affirmatively for two kinds of classes, one defined by best trigonometric approximation and the other defined by a condition on a modulus of continuity. He investigates (2) and (4) for the equivalent case of power series F. Let $E_n(F)$ denote the best approximation to $F(z)$, assumed continuous in $|z| \leqq 1$, by polynomials of degree $n-1$. His first theorem is that $E_n(F) = O(G_n)$ implies $F \epsilon A$ if and only if $\sum n^{-\frac{1}{2}}G_n$ converges. The kth modulus of continuity $\omega_k(\delta, f)$ is defined by replacing $\Delta_h F(x)$ by $\Delta_h^k F(x)$ in the usual definition. Then the author shows that under certain restrictions on $\omega(\delta)$, the condition $\omega_k(\delta, F) = O(\omega(\delta))$ implies $F \epsilon A$ if and only if $\sum n^{-\frac{1}{2}}\omega(1/n)$ converges. The same is true with no restriction (except positivity) on $\omega(\delta)$ provided that $\omega(\delta)$ is replaced by the rectified function

$$\omega_k^{**}(\delta) = \delta^k \inf_{0<\eta<\delta} \{\eta^{-k} \inf_{\eta \leq \xi \leq \pi} \omega(\xi)\}.$$

Thus (2) (and hence (1)) is answered affirmatively for the classes defined by requiring either $\sum n^{-\frac{1}{2}}G_n$ divergent or $\sum n^{-}\omega_k^{**}(1/n)$ divergent. Finally the author constructs examples to show that the answer to (4) (and hence (3)) is affirmative for both these classes. *R. P. Boas, Jr.*

Satô, Masako. Convergence of Fourier series. Proc. Japan Acad. 31 (1955), 116–118.

It is proved (with usual notations) that the two conditions

$$\varphi_x(t)-\varphi_x(t')=o(\log|t-t'|^{-1})^{-\alpha}\ \ (0<\alpha<1),\text{ as } t, t'\to 0,$$

and

$$\int_{\pi\exp\{(\log n)^{\alpha}\}/n}^{\eta}\left|\frac{\varphi_x(t)-\varphi_x(t+\pi/n)}{t}\right|dt\to 0,\text{ as } n\to\infty,$$

imply convergence of the Fourier series of $f(t)$ at $t=x$. There is also a similar result, with the log replaced by log log. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Satô, Masako. Integrability of trigonometrical series. II. Proc. Japan Acad. 31 (1955), 210–213.

The author deduces two results from theorems of Hardy and Littlewood; the second contains a theorem of the reviewer [Math. Z. 55 (1952), 183–186; MR 13, 838]. (1) If $q\geqq 2$, $p\geqq 1$, $0<\alpha<q/p-1$, $c_n\to 0$, and

$$\sum(c_{n+m}-c_{n-m})^{*q}n^{q-2}=O(m^{\alpha})$$

for $m=ks$, $k\to\infty$, then $\sum c_ne^{inx}$ is the Fourier series of a function of L^p; * denotes symmetrically decreasing rearrangement. (2) If $q\geqq 2$, $p\geqq 1$, $q'\leqq r\leqq q$, $\mu=1/r+1/q-1$, $0<\alpha<q/p-1$, $c_n\to 0$ and $\sum|c_{n+m}-c_{n-m}|^r(|n|+1)^{-\mu r}=O(m^{\alpha r/q})$ for $m=ks$, $k\to\infty$, the same conclusion holds. Theorem 1 fails if $\alpha=q/p-1>q-2$; Theorem 2 fails if $\alpha\geqq q/p-1$. *R. P. Boas, Jr.* (Evanston, Ill.).

Mohanty, R., and Nanda, M. Note on the first Cesàro mean of the derived conjugate series of Fourier series. Proc. Amer. Math. Soc. 6 (1955), 594–597.

Let $f(t)$ be an integrable function with period 2π and let

$$h(t)=(f(x+t)+f(x-t)-2f(x))/t-d.$$

The authors prove that if

$$\int_t^{\pi}|h(u)|u^{-1}du=o(\log t^{-1})\ \ (t\to 0),$$

then $t_n/\log n\to d/\pi$ $(n\to\infty)$ where t_n is the nth arithmetic mean of the derived conjugate series of Fourier series of $f(t)$. *S. Izumi* (Tokyo).

Salem, Raphaël, et Zygmund, Antoni. Sur un théorème de Piatetçki-Shapiro. C. R. Acad. Sci. Paris 240 (1955), 2040–2042.

C étant l'ensemble défini dans la Note qui suit, soit $\theta\in C$, $\theta>2$, et soit E l'ensemble du type de Cantor dont les points sont de la forme $(\theta-1)(\varepsilon_1\theta^{-1}+\varepsilon_2\theta^{-2}+\cdots)$, où les ε_j sont égaux à 0 ou 1. Piatetçki-Shapiro a démontré que si $\theta>2^n$, où n est le degré de θ, E est un ensemble d'unicité pour le développement trigonométrique [Uspehi Mat. Nauk (N.S.) 8 (1953), no. 3(55), 167–170; MR 15, 306]. Les auteurs remplacent la condition de Piatetçki-Shapiro par la condition $\theta\in C$, $\theta>2$. *S. Mandelbrojt.*

Salem, Raphaël, et Zygmund, Antoni. Sur les ensembles parfaits dissymétriques à rapport constant. C. R. Acad. Sci. Paris 240 (1955), 2281–2283.

d étant un entier positif, soit $\theta>d+1$. Considérons les points $\eta_0, \eta_1, \cdots, \eta_d$ tel que $\eta_0=0$, $\eta_{j+1}-\eta_j\geqq\theta^{-1}$, $\eta_d=1-\theta^{-1}$. En enlevant de [0, 1] les intervalles ouverts complémentaires aux intervalles fermés de longueur θ^{-1} d'origine η_j $(j=0, \cdots, d)$, on opère une "dissection", sur chacun des intervalles fermés égaux restants, "homothétique" à celle faite sur [0, 1]. En continuant cette opération indéfiniment on obtient un ensemble parfait de mesure nulle, E. Soit C la classe des entiers algébriques positifs dont tous les conjugués (autres que l'entier donné lui-même) sont, en module, strictement inférieurs à un. Salem a démontré qu'une condition nécessaire pour que E soit un ensemble d'unicité pour le développement trigonométrique est que $\theta\in C$ et que $\eta_1, \eta_2, \cdots, \eta_{d-1}$ soient des nombres algébriques du corps de θ [Trans. Amer. Math. Soc. 56 (1944), 32–39; 63 (1948), 595–598; MR 6, 47; 10, 34]. Les auteurs démontrent que ces conditions sont aussi suffisantes. *S. Mandelbrojt* (Paris).

Žak, I. E. Generalization of a theorem of V. G. Čelidze. Soobšč. Akad. Nauk Gruzin. SSR 16 (1955), 89–94. (Russian)

The author gives a number of theorems about the absolute convergence of double Fourier series. The following result is typical. Suppose that $f(x, y)$ has a Fourier series $\sum_{1,1}^{\infty}A_{mn}(x, y)$ without a constant term and terms depending on one variable only. Then $\sum A_{mn}(x, y)$ converges absolutely if $\sum_{m,n=1}^{\infty}\chi(1/m, 1/n)<\infty$, where $\chi^2(\delta_1, \delta_2)$ is equal to the upper bound of

$$\int_0^{2\pi}\int_0^{2\pi}|f(x+u, y+v)-f(x, y+v)-f(x+u, y+v)+f(u,v)|^2dx\,dy$$

for $|u|\leqq\delta_1$, $|v|\leqq\delta_2$. *A. Zygmund* (Chicago, Ill.).

Special Functions

Tricomi, Francesco G. Konfluente hypergeometrische Funktionen. Zusammenfassender Bericht. Z. Angew. Math. Phys. 6 (1955), 257–274.

In this expository paper those properties of confluent hypergeometric functions are described which are of interest to applied mathematicians. The notations and the general plan are taken from the author's books on confluent hypergeometric functions. *A. Erdélyi.*

Slater, L. J. Hypergeometric Mellin transforms. Proc. Cambridge Philos. Soc. 51 (1955), 577–589.

The author evaluates the Mellin transforms of hypergeometric functions and combinations of such functions. Her results contain as special cases many known integrals, in particular Mellin transforms involving Meijer's G-function. *A. Erdélyi* (Pasadena, Calif.).

Saran, Shanti. Transformations of certain hypergeometric functions of three variables. Acta Math. 93 (1955), 293–312.

The author defines ten hypergeometric functions of three variables, represents six of these by double or triple integrals, analogous to Euler's integral of the first kind, and obtains expansions and transformations of the six hypergeometric functions by manipulating the integrals. *A. Erdélyi* (Pasadena, Calif.).

Sharma, A., and Chak, A. M. The basic analogue of a class of polynomials. Riv. Mat. Univ. Parma 5 (1954), 325–337.

Here "basic" is used as in the theory of "basic hyper-

geometric series," not as in Whittaker's theory of basic sets of polynomials. The authors discuss the polynomials $H_n(x)$ satisfying $\{H_n(qx)-H_n(x)\}/(qx-x)=H_{n-1}(x)$ $(q\neq 0)$, giving general formulas [some of which are special cases of more general results; cf. Brenke, Amer. Math. Monthly **52** (1945), 297–301; MR **7**, 64], and a detailed discussion of some subclasses whose members satisfy an additional functional equation. *R. P. Boas, Jr.* (Evanston, Ill.).

Toscano, Letterio. Sui polinomi di Jacobi e ultrasferici. Boll. Un. Mat. Ital. (3) **10** (1955), 195–201.

The following relation between ultraspherical polynomials and Jacobi polynomials is established:

$$P_n^{(\lambda)}(\cos\theta)=e^{in\theta}P_n^{(2\lambda-1,-n-\lambda)}(z),$$

where $z(\cos\theta+i\sin\theta)=\cos\theta-3i\sin\theta$. Various applications are given, in particular to generating functions and other identities. *G. Szegö* (Stanford, Calif.).

See also: Surányi and Turán, p. 148; Brousse, p. 149; Endl, p. 149.

Ordinary Differential Equations

Krasnosel'skiĭ, M. A., and Kreĭn, S. G. Nonlocal existence theorems and uniqueness theorems for systems of ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 13–16. (Russian)

Let E denote a Banach space. For $x\in E$, $-\infty<t<\infty$, let $f(x,t)\in E$ and assume that $\|x\|<a$ and $|t|\leqq a$ imply $\|f(x,t)\|\leqq M(a)<\infty$. First a local existence theorem is stated for the equation $\dot{x}=f(x,t)$ subject to $x(t_0)=x_0$, with f continuous, $f=f_1+f_2$ with f_1 completely continuous and f_2 Lipschitz-continuous. There follow two uniqueness theorems and two theorems on the continuation of solutions over the interval $t_0\leqq t<\infty$. Suppose that $L(u)\geqq 0$ is continuous for $0\leqq u<\infty$ and that $\psi(t)\geqq 0$ is integrable for $0\leqq t_1\leqq t\leqq t_2\leqq\infty$. One existence theorem states that if there is a functional $\Psi(x)$ satisfying a uniform Lipschitz condition with $\Psi(x)\to\infty$ as $\|x\|\to\infty$, such that

$$\|f(x,t)\|\leqq L(\Psi(x))\psi(t),$$

and if $\int_X^\infty du/L(u)=\infty$ then a solution can be continued for $t_0\leqq t<\infty$. The corresponding uniqueness theorem requires that $\Psi(0)=0$, that Ψ satisfy a Lipschitz condition in a neighborhood N of the origin, that $x, y\in N$ imply $\|f(x,t)-f(y,t)\|\leqq L(\Psi(x-y))\psi(t)$, and that, for $\varepsilon>0$, $\int_0^\varepsilon du/L(u)=\infty$, and concludes that solutions with the same initial data are identical. The other theorems are similar but more complicated. Applications include systems in finite-dimensional space with various norms. *F. A. Ficken* (Knoxville, Tenn.).

Zatulovskaya, K. D. Systems of Clairaut ordinary differential equations. Uč. Zap. Ivanovsk. Pedagog. Inst. **5** (1954), 73–82. (Russian)

The author investigates a system of differential equations which he calls a Clairaut system:

$$(1)\qquad y_i=y_i'x+\varphi_i(y_1',\cdots,y_n'),\quad i=1,\cdots,n.$$

It is shown that in a certain region the general solution is a family of straight lines whose equations are obtained by replacing the derivatives in (1) by arbitrary constants. There is introduced the notion of a singular surface which is tangent to the straight-line solutions of the system (1). By means of these straight lines a field of directions is defined on the singular surface; the singular surface is filled with singular solutions.
S. A. Gal'pern and *Yu. V. Sidorov* (RŽMat **1955**, no. 5003).

Cesari, Lamberto, and Hale, Jack K. Second order linear differential systems with periodic L-integrable coefficients. Riv. Mat. Univ. Parma **5** (1954), 55–61; errata **6** (1955), 159.

Let $Y'=AY+\Phi(x,\lambda)Y$ be a differential equation, A, Φ, Y being 2×2 matrices, λ complex parameter, A real, constant; Φ complex, periodic in x of period $T=2\pi/\omega$, L-integrable in $[0,T]$, continuous in λ at $\lambda=0$ for almost all x in $[0,T]$, $\Phi(x,0)=0$, $\|\Phi(x,\lambda)\|\leqq\omega(x)$ almost everywhere in $[0,T]$ for $|\lambda|\leqq\lambda_0$, ω being L-integrable. The following statement is made: if ϱ_j are the characteristic roots of A and if either (a) $\operatorname{Re}\varrho_j<0$, or (b) $\varrho_j=\pm i\sigma$, $\sigma>0$, $2\sigma/\omega$ not integer, and if $\int_0^T \operatorname{trace}\Phi\,dx\leqq 0$ for all $|\lambda|\leqq\lambda_0$, the absolutely continuous solutions Y are bounded in $(0,+\infty)$ for $|\lambda|$ sufficiently small. Part (a) is true; part (b) is obviously false, but becomes true if it is assumed that Φ is real, which was clearly the authors' intention, as is pointed out in the errata. A second theorem says: let $y''+\psi(x,\lambda)y'+\varphi(x,\lambda)y+\sigma^2y=0$, y, σ, λ, φ, ψ real scalars, σ positive, φ, ψ periodic in x of period $T=2\pi/\omega$ and L-integrable in $[0,T]$ for all $|\lambda|\leqq\lambda_0$, continuous in λ at $\lambda=0$ for almost all $x\in[0,T]$, $\varphi(x,0)=\psi(x,0)=0$, $|\varphi|$, $|\psi|\leqq\eta(x)$, η being L-integrable, $\int_0^T\psi dx\geqq 0$; if $2\sigma/\omega$ is not an integer, the absolutely continuous solutions Y are bounded if $|\lambda|$ is small enough. *J. L. Massera* (Montevideo).

van der Merwe, J. H. Some properties of a certain linear second order differential equation. Nieuw Arch. Wisk. (3) **3** (1955), 65–71.

Let $q(x)$ be real-valued and continuous on the half-open interval $0<x\leqq 1$ and λ a complex number. The author obtains majorants, as $x\to 0$, for solutions of

$$(1)\qquad y''+(\lambda-q)y=0.$$

A typical result is the following: If $|q(x)|\leqq g(x)+h(x)$, where g, h are real continuous functions on $0<x\leqq 1$ and $z''-qz=0$ has a solutions $z=z(x)$ such that $z>0$, $z'<0$, then every solution $y=y(x)$ of (1) satisfies

$$y=O\Big(z(x)\exp\Big(\int_x^1 th(t)dt\Big)\Big)$$

as $x\to 0$. *P. Hartman* (Baltimore, Md.).

Sibirskiĭ, K. S. On conditions for the presence of a center and focus. Uč. Zap. Kišinevsk. Univ. **11** (1954), 115–117. (Russian)

In the work of N. A. Saharnikov [Prikl. Mat. Meh. **12** (1948), 669–670; MR **10**, 377] there were established necessary and sufficient conditions for a center for the equation

$$(1)\qquad \frac{dy}{dx}=-\frac{x+ax^2+(2b+\alpha)xy+cy^2}{y+bx^2+(2c+\beta)xy+dy^2},$$

where $a, b, c, d, \alpha, \beta$ are constants. One of these conditions was expressed through the coefficients of the equation obtained from (1) by rotation of the coordinate axes through some known angle. The author simplifies somewhat the writing of Saharnikov's conditions and gives the expression for the condition mentioned above in terms of the coefficients of (1).
V. P. Basov (RŽMat **1955**, no. 5012).

Krasovskiĭ, N. N. On stability of motion in the critical case of a single zero root. Mat. Sb. N.S. 37(79) (1955), 83–88. (Russian)

Consider a system (1) $\dot{x}=X(x)$, where x, X are $(n+1)$-vectors, $X(0)=0$ and X is of class C' at the origin. Suppose also that the matrix A of the first-degree terms has one characteristic root zero and the rest with negative real parts. Let $J=\|\partial X_i/\partial x_j\|$. Theorem: If in a neighborhood of the origin (origin excepted) the characteristic roots of J have their real parts all negative [some positive] then the origin is asymptotically stable [is unstable] for (1).

This question was already dealt with by Lyapunov [Problème général de la stabilité du mouvement, Princeton, 1947; MR 9, 34] who first reduced (1) to a special form. His condition reads decidedly differently. The present author's starting point is Lyapunov's reduced system. His condition offers the decided advantage that it is expressed directly in terms of (1) without requiring a preliminary transformation. [Additional references: Krasovskiĭ, Dokl. Akad. Nauk SSSR (N.S.) 93 (1953), 965–967; MR 15, 795; Četaev, Stability of motion, Gostehizdat, Moscow, 1946]. *S. Lefschetz* (Princeton, N.J.).

Pliss, V. A. Necessary and sufficient conditions for stability in the large for a system of n differential equations. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 17–18. (Russian)

Erugin has given conditions for the stability in the large of the origin for a planar autonomous system of differential equations [see Prikl. Mat. Meh. 14 (1950), 459–512; MR 12, 412]. His treatment leans heavily upon the fact that the dimension is two. In the present note his theorem is extended to any dimension. Consider

(1) $$\dot{x}=X(x),$$

where x, X are n-vectors, $X(0)=0$, and X is continuous and satisfies uniqueness conditions for all x. Let L denote the hyperplane $L(a,x)=a_1x_1+\cdots+a_nx_n=0$. The path $M(t)$ is said to cross L at $M(t_1)$ at time t_1 if there are times $t_0<t_1\leq t_2<t_3$ such that on the path $L(a,x)=0$ for $t\in[t_1,t_2]$ while $L(a,x)\neq 0$ for $t\in(t_0,t_1)$ or (t_2,t_3) with opposite signs on the two intervals. Theorem: Necessary and sufficient conditions for asymptotic stability in the large of the origin are: I. The origin must be the only critical point of (1), and it must be stable. II. There exists an L such that: (a) if there corresponds to a path Γ a value T such that, for $t>T$, Γ does not intersect L, then $x(t)\to 0$ as $t\to+\infty$; (b) there is a function $v(x)$ defined on L such that $v(0)=0$, $v(x)>0$ for $x\in L$—the origin, $v(x)\to+\infty$ with x on L; (c) if the path $M(t)$ meets L at two points $M(t_1)$, $M(t_2)$, $t_1<t_2$ then $v(M(t_1))>v(M(t_2))$.

Necessity is proved by reference to Barbašin and Krasovskiĭ [ibid. 18 (1954), 345–350; MR 15, 957]. In outline sufficiency is established as follows: Let the path $M(t)$ cut L at times $\{t_k\}\to+\infty$. Owing to IIbc, the points $M(t_k)$ are in a bounded neighborhood $v\leq v(M(t_1))$. Hence $M(t)$ has an ω limit-point A in that set. By I and II one may show that A is the origin and from I that it is the only ω-limit point of $M(t)$, and sufficiency follows.

The theorem is applied to show that: (a) the origin is stable in the large for

(2) $$\dot{x}=y,\ \dot{y}=x-ay,\ \dot{z}=-\varphi(y)-bx,$$

where a, b are positive constants, $\varphi(0)=0$, φ is continuous and satisfies a uniqueness condition and $\varphi(y)/y>b/a$ for $y\neq 0$. This generalizes a result of Barbašin [ibid. 16 (1952), 629–632; MR 14, 376]; (b) the origin is stable in the large for

(3) $$\dot{x}=y,\ \dot{y}=x-f(x),\ \dot{z}=-by-cz,$$

where b, c are positive constants such that $c^2-4b>0$, and $f(0)=0$, f is continuous, satisfies a uniqueness condition and $xf(x)>0$ for $x\neq 0$. *S. Lefschetz* (Princeton, N.J.).

Vinograd, R. È. Remark on the critical case of stability of a singular point in the plane. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 209–212. (Russian)

Let $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ be a real autonomous differential system in an open neighborhood U of the origin which is an isolated critical point (that is, $P(0,0)=Q(0,0)=0$). Also $P(x,y)$, $Q(x,y)$ are in $C^{(1)}$ in U. Let $\alpha_1(x,y)$ and $\alpha_2(x,y)$ be the real parts of the roots of the quadratic equation $\lambda^2-S(x,y)\lambda+\Delta(x,y)=0$ where $S(x,y)=P_x+Q_y$ and $\Delta(x,y)=P_xQ_y-P_yQ_x$. Then the author investigates the "critical case" where $\alpha_1(x,y)\alpha_2(x,y)=0$ only in a nowhere dense subset of U. The conclusion is that (a) if $\alpha_1\leq 0$, $\alpha_2\leq 0$ in U, then the origin is stable; (b) if $\alpha_1\geq 0$, $\alpha_2\geq 0$ in U, the origin is completely unstable; and (c) if $\alpha_1\alpha_2\leq 0$ in U, then the origin is semi-stable. [Cf. A. Šestakov, same Dokl. (N.S.) 65 (1949), 139–142; MR 10, 535.] *L. Markus.*

Krasnosel'skiĭ, M. A., and Kreĭn, S. G. On the principle of averaging in nonlinear mechanics. Uspehi Mat. Nauk (N.S.) 10 (1955), no. 3(65), 147–152. (Russian)

Given the system

(1) $$\dot{x}=\varepsilon X(x,t)$$

with x, X n-vectors and x varying in a bounded domain D of E^n, suppose that for every x in D

(2) $$\lim_{T\to+\infty}\frac{1}{T}\int_0^T X(x,t)dt=X_0(x)$$

exists. Take now the system

(3) $$\dot{y}=\varepsilon X_0(y)$$

and let $x(t)$, $y(t)$ be solutions of (1) and (3) such that $x(0)=y(0)=x_0$. Bogoliubov proved [On some statistical methods in mathematical physics, Akad. Nauk Ukrain. SSR, 1945; MR 8, 37] the following theorem. Let $X(t,x)$ be bounded in D and satisfy there a Lipschitz condition with constant independent of x, t. Let also the limit (2) exist for every x in D. Suppose finally that $y(t)$ is known for $\varepsilon=1$ and $t\in[0-T]$ and together with a certain neighborhood does lie in D. Then, given $\eta>0$, there exists $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$, $x(t)$ as defined above is in modulus within an η-neighborhood of $y(t)$ on $t\in[0-T/\varepsilon]$.

It was shown by Gikhman [Ukrain. Mat. Ž. 4 (1952), 215–218 (unavailable for review)] that the above theorem is a ready consequence of a theorem on the continuous dependence of the solution of a differential equation on a parameter. Gikhman leaned heavily upon the Lipschitz condition. His result is proved here under much more general conditions, and this extends considerably the reach of the theorem of Bogoliubov. *S. Lefschetz.*

Furuya, Shigeru. Periodic solutions of a nonlinear differential equation. Comment. Math. Univ. St. Paul. 4 (1955), 47–51.

By means of the perturbation method the author studies the periodic solutions of the equation

(1) $$\ddot{x}+b\dot{x}+x+(a-cx^2)x\cos 2t+ex^3=0$$

for small values of a, b, c, and e. Minorsky [J. Franklin

Inst. **254** (1952), 21–42; MR **14**, 170] has investigated the periodic solutions of (1) for certain special cases in which one or more of the parameters is zero. The present author treats the general problem, dividing the cases up into five classes defined by inequalities; e.g. one class is defined by $|a|/b>2$ and $|c|/|e|>3/2$. For each class the possible number of periodic solutions, their stability and some bounds on their amplitudes are determined. The cases in which equalities occur, e.g. $|a|/b=2$, are not considered. No analytical details are given but the results can evidently be obtained by fairly elementary considerations.
C. E. Langenhop (Ames, Iowa).

★**Haag, Jules. Les mouvements vibratoires. Tome 2.** Presses Universitaires de France, Paris, 1955. viii+253 pp. 1600 francs.

The first volume of this work has been reviewed previously [1952; MR **15**, 223]. Whereas that volume was concerned with vibrations of systems having one degree of freedom, the present volume deals with vibrations of systems having several degrees of freedom. For the most part, the systems under consideration are linear. There are chapters devoted to generalities concerning the oscillations of a linear system with n degrees of freedom, to problems concerning stability, to the effects of various kinds of coupling between degrees of freedom, and to the theory of wave filters. In addition, there are highly interesting chapters concerning various oscillatory motions of the automobile, of the bicycle and tricycle, and of the gyroscope. Four chapters devoted to various other topics, including two devoted to synchronization problems, complete the book.

In some places the author's methods and style make the reader's task unnecessarily difficult. Thus, in the chapter on wave filters, many results which are ordinarily obtained in thoroughly direct and elementary ways are here obtained by means of a complicated algebraic method which is by no means easy to follow. Again, in the chapters dealing with the automobile, bicycle and tricycle, overly brief statements of the physical situations under consideration, and rather inadequate diagrams, make for laborious reading. However, despite these defects, the book is to be highly recommended, as one which contains a large amount of important and interesting theory, competently expounded. *L. A. MacColl*

Miščenko, E. F., and Pontryagin, L. S. Periodic solutions of systems of differential equations near to discontinuous ones. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 889–891. (Russian)

The system under discussion is

$$\varepsilon\dot{x}=f(x, y),\ \dot{y}=g(x, y), \tag{1}$$

where x and y are a p- and a q-vector and ε is small. Let the system

$$f=0,\ \dot{y}=g \tag{2}$$

have a "discontinuous" periodic solution γ of a type familiar in investigations of van der Pol's equation. The authors determine the periodic solutions of (1) close $O(\varepsilon)$ to γ and calculate their period. The qualitative analysis of the situation was already made by Železcov and Rodygin [same Dokl. (N.S.) **81** (1951), 391–394]. An analogous determination for van der Pol's equation was already established by Dorodnytsin [Prikl. Mat. Meh. **11** (1947), 313–328; MR **9**, 144]. [Remark by the reviewer. The authors are not aware of a number of related "western" contributions, the most notable being the work of Jules Haag, Les mouvements vibratoires, t. 1, 2, Presses Univ. de France, Paris, 1952, 1955; MR **15**, 223; and the preceding review]. *S. Lefschetz* (Princeton, N.J.).

★**Amerio, Luigi, Bertolini, Italo, Danusso, Arturo, De Marchi, Giulio, Finzi, Bruno, Krall, Giulio, Masotti, Arnaldo, Pastori, Maria, e Sesini, Ottorino. Lezioni sulle vibrazioni meccaniche.** Libreria Editrice Politecnica Cesare Tamburini, Milano, 1952. iii+308 pp. 2800 Lire.

This volume contains the substance of a short course of lectures which was given, in 1949, in a seminar in Milan. The purpose of the course was to bring together mathematicians, physicists and engineers who are interested in the theory of vibrations, and to develop the parts of the theory entering into the various applications from a unified point of view. Although the editor states that the published version of the lectures is somewhat different from that given in the course, it appears to the reviewer that the course must have served its purpose admirably.

Of the eleven chapters, seven are rather definitely theoretical in character. These deal with vibrations of systems having one degree of freedom, Fourier series, vibrations of systems having n degrees of freedom, linear differential equations, vibrations of rotating shafts, vibrations of continua, and vibrations due to aerodynamic forces, respectively. The remaining four chapters are more descriptive. These discuss vibrations of machines, vibrations of buildings, oscillations in hydraulic systems, and fatique of materials, respectively.

The book is very well printed, and several of the chapters are accompanied by rather extensive bibliographies. *L. A. MacColl* (New York, N.Y.).

Bloh, Z. Š. Some estimates of the quality of regulation from the frequency characteristics. Avtomat. i Telemeh. **16** (1955), 258–268. (Russian)

Estimates for quantities that measure the quality of regulation of linear servo systems are obtained from known formulas which relate the performance of the system to the transfer function $\Phi(i\omega)$; $\mathfrak{R}(\omega)$ and $\mathfrak{J}(\omega)$ are defined to be the real and imaginary parts, respectively, of $\Phi(i\omega)$. Let $\varphi(t)$ be the response of the system to a unit step function. The basic relations used are:

$$\Phi(z)=L(\varphi'(t))=\int_0^\infty \varphi'(t)e^{-zt}dt, \tag{1}$$

$$\varphi(t)=\frac{2}{\pi}\int_0^\infty \frac{\mathfrak{R}(\omega)}{\omega}\sin \omega t\, dt, \tag{2}$$

$$\varphi(t)=1+\frac{2}{\pi}\int_0^\infty \frac{\mathfrak{J}(\omega)}{\omega}\cos \omega t\, dt, \tag{3}$$

where $\Phi(i\omega)=\mathfrak{R}(\omega)+i\mathfrak{J}(\omega)$. It is assumed that $\varphi(\infty)=1$; i.e. the servo has a zero static error.

Necessary conditions are given that $\varphi(t)$ be monotonic. It follows from (1) that (4) $M(\omega)=|\Phi(i\omega)|\leqq\int_0^\infty|\varphi'(t)|dt$ and, if $\varphi(t)$ is monotonic, this estimate becomes $M(\omega)\leqq 1$. It is also shown that $|S(\omega)|\leqq|S(0)|$, where $S(\omega)=\mathfrak{J}(\omega)/\omega$, is a necessary condition that $\varphi(t)$ be monotonic. Estimates from below of the "over-regulation" of the system are obtained under the assumption that $\varphi(t)$ has a finite number of maxima and minima. For example, when φ has a single maximum, it follows from (4) that $\varphi_{\max}\geqq\frac{1}{2}(1+\max M(\omega))$. Under the assumption that $\mathfrak{R}(\omega)$ has a finite number of maxima and minima in the main frequency interval, estimates from above are ob-

tained on the "over-regulation". These estimates are used to obtain estimates on the time of regulation.

The application of some of these results are illustrated for a single-loop servo; $\Phi(i\omega)=w_0(i\omega)/[1+w_0(i\omega)]$. This is done by plotting the Nyquist diagram of the feedback transfer function. *J. P. LaSalle* (Notre Dame, Ind.).

Bautin, N. N. The dynamical theory of clock movements without constructive stopping of the escape wheel (instantaneous impulse). Inžen. Sb. **21** (1955), 3–31. (Russian)

Mathematically speaking the scheme gives rise to a pair of sets of linear differential equations with constant coefficients with passage from one set to the other across a point of discontinuity. Its discussion turns upon the study of a linear mapping of a three-space of initial values into itself. The resulting formulas, including those for stability, are far too complicated and too technical to warrant description [see Bautin, Dokl. Akad. Nauk SSSR (N.S.) **61** (1948), 17–20; **65** (1949), 279–282; MR **10**, 409; 748; also Andronov and Naimark, ibid. **51** (1946), 17–20; MR **8**, 101]. *S. Lefschetz* (Princeton, N.J.).

Andronov, A. A., and Bautin, N. N. The theory of course stabilization of a neutral airplane by means of an autopilot with a constant-speed servomotor. I. Case of absence of a zone of insensitivity. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 3, 3–32. (Russian)

The present work considers (under the usual simplifying hypothesis) the theory of course stabilization of an aircraft (without a stabilizing moment) by means of an autopilot with a constant-speed servomotor for the course controls. It is assumed that the automatic installation has a feed-back and also an artificial damping device. This work covers also the theory of other regulating systems whose mathematical model consists of the same type of differential equations as those considered here. The equations of motion of the given system are

$$\ddot{\varphi}+M\dot{\varphi}=-N\eta,\ \dot{\eta}=F(\psi),\ \psi=\varphi+\beta\dot{\varphi}-a^{-1}\eta.$$

Here φ is the attack angle of the aircraft, η is the turning angle of the rudder. The positive numbers M and N are constants which describe the natural damping, and the steering mechanism; ψ is the variable of the servomotor which controls the rudder; the positive number β is the coefficient of artificial damping, while the positive number a^{-1} is the coefficient of the feed-back. The function $F(\psi)$ is called the characteristic of the servomotor. For the given case, when the servomotor has a constant speed and a zone of insensitivity,

$$F(\psi)=\begin{cases} K \text{ if } & \psi>\psi_0,\\ 0 \text{ if } & |\psi|<\psi_0,\\ -K \text{ if } & \psi<-\psi_0,\end{cases}$$

where K is the constant speed of the servomotor and $2\psi_0$ is the width of the symmetric zone of insensitivity.

Formulas and diagrams are constructed which permit a quick determination of the existence or absence of auto-oscillations, and give an estimate of the region in which the initial conditions must lie if the stabilization process is to converge. *H. P. Thielman* (Ames, Iowa).

Moser, Jürgen. Singular perturbation of eigenvalue problems for linear differential equations of even order. Comm. Pure Appl. Math. **8** (1955), 251–278.

The author examines the asymptotic behavior, for $\varepsilon\to 0$, of the eigenvalues λ and eigensolutions u of the problem (a): $(P+\varepsilon Q)u=\lambda u$ in $a\leqq x\leqq b$ with homogeneous linear boundary conditions independent of ε at $x=a$, $x=b$. Here P and Q are linear ordinary differential operators, Q being of higher order than P. The asymptotic behavior of solutions of equations $(P+\varepsilon Q)u=0$ has been studied most generally by Turrittin [Contributions to the theory of nonlinear oscillations, v. II, Princeton, 1952, pp. 81–116; MR **14**, 377]. In this paper P and Q are non-singular of even order with infinitely differentiable coefficients.

The boundary conditions are reduced to

$$A_r u=u^{(\alpha_r)}(a)+\sum_{s<\alpha_r} a_{rs}u^{(s)}(a)=0,$$

$$B_r u=u^{(\beta_r)}(b)+\sum_{s<\beta_r} a_{rs}u^{(s)}(b)=0,$$

where $u^{(s)}=d^su/dx^s$; $r=0, 1, \cdots, n$; $0\leqq\alpha_1<\cdots<\alpha_n<2n$; $0\leqq\beta_1<\cdots<\beta_r<2n$. Under some conditions on α_r, β_r, it is proved that if $\lambda=\lambda_0$, $u=U_0(x)$ are a simple eigenvalue and eigensolution of problem (b): $Pu=\lambda u$, $A_ru=0$, $B_ru=0$, $r=1, 2, \cdots, m$, then for small $\varepsilon>0$ there exists in a neighborhood of $\lambda=\lambda_0$ a uniquely determined eigenvalue $\lambda(\varepsilon)$ and a corresponding eigensolution U belong to (a), such that λ and U have asymptotic expansions up to every order in $\eta=\varepsilon^{1/2(n-m)}\geqq 0$. The leading terms are λ_0 and $U_0(x)$ respectively. The expansion for $U(x,\varepsilon)$ is valid for $a+\delta\leqq x\leqq b-\delta$, $0<\delta<\frac{1}{2}(b-a)$ and can be differentiated $2n-1$ times. If λ_0 is real, $\lambda(\varepsilon)$ and U are real.

For self-adjoint problems it is shown that the conditions on α_r, β_r are satisfied for certain boundary conditions, and that $\lim_{\varepsilon\to 0}\lambda_n(\varepsilon)=\lambda_n(0)$ where $\lambda_n(\varepsilon)$, $\varepsilon>0$, $n=0, 1, 2, \cdots$, are all the eigenvalues of (a) and $\lambda_n(0)$, $n=0, 1, 2, \cdots$ are all the eigenvalues of (b). *C. S. Morawetz* (New York, N.Y.).

Kolodner, Ignace I. Heavy rotating string — a nonlinear eigenvalue problem. Comm. Pure Appl. Math. **8** (1955), 395–408.

The author's abstract is: "Rotations of a heavy string with one endpoint free are considered. According to the linear theory, a string of a given length can rotate only at certain eigenvelocities of rotation ω_n which form a discrete spectrum. It is shown that according to the more accurate non-linear theory, a string can rotate at any velocity $\omega>\omega_1$, and that for each ω in the range $\omega_n<\omega\leqq\omega_{n+1}$ there are exactly n distinct modes of rotation."

The string in question is hanging under gravity. The author reduces the mechanical problem to a consideration of the equation

$$v''+(v^2+x^2)^{-\frac{1}{2}}v=0,\ v(0,a)=0,\ v'(0,a)=a\geqq 0,\ x\geqq 0,\ v\in C_2.$$

The bulk of the paper is concerned with this equation. It is shown that the solution exists and is unique, and that it and its derivative have infinitely many interlaced zeros. Several properties of these zeros are established. These enable the author to prove the last statement in his abstract above. *E. Pinney* (Berkeley, Calif.).

Zadiraka, K. V. Construction of upper and lower estimates for eigenvalues of a one-dimensional self-adjoint boundary problem of even order. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 681–684. (Russian)

The problem considered is of the form

$$(-1)^k d^{2k}y/dx^{2k}+q(x)y=\lambda p(x)y,$$

$$\sin\alpha_s\, y^{(s)}(0)+\cos\alpha_s\, y^{(2k-1-s)}(0)=0,$$

$$\sin\beta_s\, y^{(s)}(1)+\cos\beta_s\, y^{(2k-1-s)}(1)=0.$$

In developing the method, the author obtains a sequence of upper and lower functions for the equation

$$d^{2k}y/dx^{2k}+Q(x)y=0$$

with given initial conditions, where Q is to be identified eventually with $q-\lambda p$. To solve the required problem exactly one could form first k solutions of the differential equation, each satisfying the conditions at the initial boundary, and select λ so that a certain kth order determinant vanishes. The bounds are obtained by introducing upper and lower functions into the determinant.

A. S. Householder (Oak Ridge, Tenn.).

Newton, R. G., and Jost, R. The construction of potentials from the *S*-matrix for systems of differential equations. Nuovo Cimento (10) **1** (1955), 590–622.

Systems of equations of the type of the Schrödinger equation for the *S*-states are considered. The potential, if it exists, is determined uniquely by the *S*-matrix, the eigenvalues, and a symmetric, positive semi-definite matrix associated with each eigenvalue. The Gelfand-Levitan method is used. It is necessary to factor a matrix given along the real axis into a product of two matrices, one analytic in the upper half-plane and the other in the lower half-plane. A procedure of Plemelj is used.

N. Levinson (Cambridge, Mass.).

Kay, Irvin W., and Moses, Harry E. The determination of the scattering potential from the spectral measure function. II. Point eigenvalues and proper eigenfunctions. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-19** (1955), i+22 pp.

From the author's introduction: "In part I of this paper [Kay, same Rep. No. **EM-74** (1955); MR **16**, 1113] we restricted ourselves to the consideration of weight operators which led to Hamiltonians H whose spectra were identical to the spectrum of a given unperturbed Hamiltonian H_0. In this part of the paper we propose to show how we may choose weight operators which lead to Hamiltonians H having spectra different from that of H_0. To make the discussion more concrete we shall take the case where H_0 has a purely continuous spectrum extending from 0 to ∞ and where H has a spectrum which has a continuous part which coincides with the spectrum of H_0, and, in addition, has negative point eigenvalues."

N. Levinson (Cambridge, Mass.).

See also: Bellman, p. 151.

Partial Differential Equations

***Petrowski, I. G. Vorlesungen über partielle Differentialgleichungen.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1955. 296 pp. DM 17.00.

Translation of Petrovskiĭ's Lekcii ob uravneniyah s častnymi proizvodnymi, 2d ed. [Gostehizdat, Moscow, 1953; MR **16**, 133].

Samuèl', O. I. Necessary conditions for group separation of variables in the complete integral of a partial differential equation. Trudy Sredneaz. Univ. **1954**, no. 36, Mat. Nauk. **7**, 65–73. (Russian)

There are established some necessary conditions under which the complete integral of the equation

$$F(x_1, \cdots, x_n, \partial z/\partial x_1, \cdots, \partial z/\partial x_n)=0$$

allows group separation of variables:

$$z=\varphi_1(x_1, \cdots, x_{n_1})+\varphi_2(x_{n_1+1}, \cdots, x_{n_2})+\cdots +\varphi_k(x_{n_{k-1}+1}, \cdots, x_n).$$

Z. I. Halilov (RŽMat **1955**, no. 5028).

Ionescu-Cazimir, V. La définition des variétés caractéristiques pour certains systèmes d'équations à dérivées partielles. Com. Acad. R. P. Române **1** (1951), 323–325. (Romanian. Russian and French summaries)

The systems of partial differential equations

$$(*)\qquad \frac{\partial Q_i}{\partial x_\mu}=\sum_j a_{\mu ij}\frac{\partial Q_j}{\partial x_0}$$

and

$$(**)\qquad \frac{\partial Q_i}{\partial x_\mu}=\sum_j a_{\mu ij}\frac{\partial Q_j}{\partial x_{n+1}}+\sum_j b_{\mu ij}\frac{\partial Q_j}{\partial x_{n+2}},$$

where the $a_{\mu ij}$ are constants, are dealt with from the point of view of characteristics; thus the characteristic curves of the system (*) are shown to be integrals of a Monge equation, while the characteristic surfaces for the more general system (**) are the integrals of an algebraic equation involving the exterior products of the differentials involved. *E. F. Beckenbach* (Los Angeles, Calif.).

Moisil, Gr. C. La définition des systèmes différentiels adjoints pour quelques systèmes qui ne sont pas de type Cauchy-Kowalewska. Com. Acad. R. P. Române **1** (1951), 327–329. (Romanian. Russian and French summaries)

It is pointed out that the system of partial differential equations (*) in the preceding review, and the adjoint system

$$\frac{\partial \psi_j}{\partial x_\mu}=\sum_i a_{\mu ij}\frac{\partial \psi_i}{\partial x_0},$$

lead to the reciprocity formulas

$$\int \Big(\sum_i \psi_i\varphi_i dx_0+\sum_{\mu,i,j} a_{\mu ij}\psi_i\varphi_j dx_\mu\Big)=0.$$

The system (*) expresses the condition of monogeneity, in the sense of Fedorov, of a function of the hypercomplex variable

$$\eta=x_0+\sum_\mu x_\mu A_\mu,\quad A_\mu=\{a_{\mu ij}\}.$$

Similarly, the more general system (**) expresses the condition of monogeneity of a function of two hypercomplex variables. *E. F. Beckenbach.*

Ionescu-Cazimir, Viorica. Sur les caractéristiques de certains systèmes d'équations aux dérivées partielles. III. Com. Acad. R. P. Române **1** (1951), 909–914. (Romanian. Russian and French summaries)

For the solutions of the systems (*) and (**) discussed in the two preceding reviews, the author now treats the hypersurfaces of discontinuity. Several examples are given; thus for the system $\partial P/\partial x=\partial Q/\partial y=\partial R/\partial z$, $\partial P/\partial y=\partial Q/\partial z=\partial R/\partial x$, $\partial P/\partial z=\partial Q/\partial x=\partial R/\partial y$ the hypersurfaces of discontinuity are the integrals of the system $\partial V/\partial x=\alpha_h\partial V/\partial y=\alpha_h^2\partial V/\partial z$ $(h=1, 2, 3)$, where $\alpha^3=1$.

E. F. Beckenbach (Los Angeles, Calif.).

Moisil, Gr. C. Sur les intégrales exponentielles des systèmes d'équations aux dérivées partielles à une fonction inconnue. Com. Acad. R. P. Române **2** (1952), 307–309. (Romanian. Russian and French summaries)

The author proves the following two theorems.

(1) If a system of linear partial differential equations with constant coefficients in a single unknown function $\varphi(x_1, \cdots, x_n)$ has no solution of the form

$$\varphi=\exp(\xi_1x_1+\cdots+\xi_nx_n),$$

it has no non-vanishing solution. (2) If the system admits only a finite number of linearly independent exponential solutions, then the general solution depends on a finite number of arbitrary constants. The notations used are not entirely clear to the reviewer, yet seem to suggest, that only homogeneous equations are considered.

E. Grosswald (Philadelphia, Pa.).

Levin, J. J. First order partial differential equations containing a small parameter. J. Rational Mech. Anal. 4, (1955) 481–501.

The author considers the boundary-value problem

$$\varepsilon\left\{a_1(x, y, u, \varepsilon)\frac{\partial u}{\partial x}+a_2(x, y, u, \varepsilon)\frac{\partial u}{\partial y}\right\}=b(x, y, u, \varepsilon)$$

satisfying $u=f_3(\tau)$ on $x=f_1(\tau)$, $y=f_2(\tau)$ $(\tau_0\leqq\tau\leqq\tau_1)$. Denoting the solution by $u(P, \varepsilon)$, the problem considered is to investigate $\lim_{\varepsilon\to 0} u(P, \varepsilon)$ and its relation to the surface $b(x, y, u(x, y), 0)=0$. A preliminary transformation is made to yield the equivalent problem

$$\varepsilon\left\{\frac{\partial v}{\partial x}+a(x, y, v, \varepsilon)\frac{\partial v}{\partial y}\right\}=b(x, y, v, \varepsilon),\ v=\gamma(t) \text{ on } x=0,\ y=t \quad (t_0\leqq t\leqq t_1),$$

where the following conditions are assumed: a) Q, R are the regions

$$Q\colon\ 0\leqq x\leqq Z(y)\ \ (t_0-\lambda\leqq y\leqq t_1+\lambda),$$
$$R\colon\ (x, y)\in Q,\ |v|<\delta_0,\ 0\leqq\varepsilon\leqq\varepsilon_0,$$

where $Z(y)\in C'(y)$; λ, δ_0, $\varepsilon_0>0$, are constants; b) $b(x, y, 0, 0)\equiv 0$, $b_v(x, y, 0, 0)\leqq-\mu<0$, in Q; c) a, $b\in C^2(x, y, v)$ in R, and the derivatives are continuous in ε as well; d) $a(x, y, 0, 0)\equiv 0$ in Q; e) $\gamma(t)\in C'(t)$ $(t_0\leqq t\leqq t_1)$, and $(0, t, \gamma(t), 0)$ lies in the interior of R for $t_0\leqq t\leqq t_1$.

Then, in any region D formed by $y=0$, $t_0\leqq t\leqq t_1$, two straight lines from the ends of this interval which intersect $x=Z(y)$, but not each other in Q, and the intersected arc on $Z(y)$, the unique solution $v=H(x, y, \varepsilon)$ of the boundary-value problem (together with its first derivatives) tends uniformly to zero in the subregion of D for which $x\geqq x_0>0$. It is to be noted that $v\equiv 0$ is the solution of the "reduced" equation $b(x, y, v, 0)=0$ in Q.

G. E. Latta.

Oliveri, E. Condizione caratteristica perchè un sistema di equazioni differenziali lineari sia canonico. Matematiche, Catania 9 (1954), 148–153.

The theorem stated and proved by the author is equivalent to the statement of the almost obvious fact that the integrability conditions of the equations

$$\partial H/\partial q_h=-\sum_{k=1}^{n}(a_{2h-1,2k-1}q_k+a_{2h-1,\,2k}p_k),$$
$$\partial H/\partial p_h=+\sum_{k=1}^{n}(a_{2h,2k-1}q_k+a_{2h,2k}p_k)\quad(h=1, \cdots, n),$$

reduce to the form $a_{ij}=(-1)^{i+j}a_{ji}$ $(i, j=1, \cdots, 2n)$.

D. C. Lewis (Pacific Palisades, Calif.).

Ezra, Jacques. Sur certains systèmes d'équations aux dérivées partielles quasi linéaires. II. C. R. Acad. Sci. Paris 241 (1955), 537–539.

[For I see same C.R. 240 (1955), 270–272; MR 16, 592; in that review the first "$P=Q$" should read "$P\neq Q$".] Let D be a domain bounded by a sufficiently smooth curve C. Let $f^i\equiv f^i(x, y; x_0, y_0)$ $(i=1, 2)$ have two continuous derivatives in D. If u and v satisfy the equations

$$\frac{\partial}{\partial x}X^i(x, y, u, v)=\frac{\partial}{\partial y}Y^i(x, y, u, v)+Z^i(x, y, u, v),$$

then Green's formula yields

$$\int_D(f^i{}_{vy}Y^i-f^i{}_{vx}X^i-f_v{}^iZ^i)dx\,dy+\int_C f_v{}^i(Y^idx+X^idy)=0.$$

This relation is now regarded as an integral equation for Y^i with kernel $f^i{}_{vv}$ which may be solved for

$$Y^i(x_0, y_0, u(x_0, y_0), v(x_0, y_0)).$$

This method is applied to solve $u_x-v_y=A(x, y, u, v)$, $v_x+u_y=B(x, y, u, v)$ subject to the boundary condition $lu+mv=r$ with l, m, r, smooth functions on C. The functions f^i are chosen so as to make Schauder's fixed-point theorem applicable.

F. A. Ficken.

Lukomskaya, M. A. On the set of solutions of a system of homogeneous differential equations. Uč. Zap. Belorussk. Gos. Univ. Ser. Fiz.-Mat. 1953, no. 15, 41–45. (Russian)

It is proved that the system of equations

$$(1)\qquad \begin{aligned} c_1\frac{\partial u}{\partial x}+d_1\frac{\partial v}{\partial x}&=a_1\frac{\partial u}{\partial y}+b_1\frac{\partial v}{\partial y},\\ c_2\frac{\partial u}{\partial x}+d_2\frac{\partial v}{\partial x}&=a_2\frac{\partial u}{\partial y}+b_2\frac{\partial v}{\partial y},\end{aligned}$$

where a_i, b_i, c_i, d_i are sufficiently smooth functions such that

$$\frac{\partial c_i}{\partial x}=\frac{\partial a_i}{\partial y},\quad \frac{\partial d_i}{\partial x}=\frac{\partial b_i}{\partial y}\quad(i=1, 2),$$

$$\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2+\begin{vmatrix}c_1 & d_1\\ c_2 & d_2\end{vmatrix}^2+\begin{vmatrix}a_1 & d_1\\ a_2 & d_2\end{vmatrix}^2\not\equiv 0,$$

$$a_1d_2=b_2c_1,\ a_1(d_1-c_2)=c_1(b_1-a_2),\ b_2(d_1-c_2)=d_2(b_1-a_2),$$

is closed with respect to Σ-integration [Bers and Gelbart, Trans. Amer. Math. Soc. 56 (1944), 67–93; MR 6, 86]. This means that in order that the function $f_1(z)=u_1+iv_1$ be a solution of (1) it is necessary and sufficient that

$$f_1(z)=\int_{z_0}^{z}f(z)d_\Sigma z=\int_{(x_0,y_0)}^{(x,y)}(a_2u+b_2v)dx+(c_2u+d_2v)dy$$
$$+i\int_{(x_0,y_0)}^{(x,y)}(a_1u+b_1v)dx+(c_1u+d_1v)dy,$$

where $f(z)=u+iv$ is a solution of (1).

G. N. Položiĭ (RŽMat 1953, no. 747).

Lukomskaya, M. A. On some systems of homogeneous linear differential equations. Uč. Zap. Belorussk. Gos. Univ. Ser. Fiz.-Mat. 1954, no. 16, 56–64. (Russian)

The form of the system and the hypotheses are the same as in the work of the author reviewed above; in addition it is required that $(a_1b_2-a_2b_1)(c_1d_2-c_2d_1)\neq 0$. The Σ-derivative of the function $f(z)=u(x, y)+iv(x, y)$ is defined by the formula

$$f_\Sigma^{[1]}(z)=\left[\left(b_2\frac{\partial v}{\partial x}-b_1\frac{\partial u}{\partial x}\right)+i\left(a_1\frac{\partial u}{\partial x}-a_2\frac{\partial v}{\partial x}\right)\right](a_1b_2-a_2b_1)^{-1};$$

$f_\Sigma^{[n]}(z)$ is the result of n-fold iterated Σ-differentiation. Some theorems are proved which were formulated earlier by the author [Dokl. Akad. Nauk SSSR (N.S.) 73 (1950), 885–888; MR 12, 604]. Among them are an analogue of

Morera's theorem, Weierstrass's theorem on the uniform limit of analytic functions, an estimate of a Σ-monogenic function (proved under the hypothesis of analyticity in x and y of the coefficients of the system and of the studied solution) $|f_\Sigma^{(n)}(z_0)|<c^n n!$ $(n=1, 2, \cdots)$, and also the equation $f(z)\equiv 0$ for a Σ-monogenic function as a consequence of the equations $f_\Sigma^{(n)}(z_0)=0$ $(n=0, 1, \cdots)$ (under the same assumptions). Some results are formulated without proof. There is also considered the system

$$\sum_{k=1}^{n}\left(a_{jk}\frac{\partial u_k}{\partial x}-b_{jk}\frac{\partial u_k}{\partial y}\right)=0 \quad (j=1, \cdots, n)$$

with sufficiently smooth coefficients satisfying the conditions $\partial a_{jk}/\partial x=\partial b_{jk}/\partial y$ $(j, k=1, \cdots, n)$, $\det a_{jk}\cdot\det b_{jk}\neq 0$. The Σ-integral of the solution $(u_1, \cdots, u_n)$ is defined as the vector with components

$$U_j=\int_{(x_0,y_0)}^{(x,y)}\sum_{k=1}^{n}(b_{n+1-j,k}u_k dx+a_{n+1-j,k}u_k dy).$$

Conditions of closure of the system relative to Σ-integration take the form

$$\sum_{k=1}^{n}(a_{jk}b_{n+1-k,l}-b_{jk}a_{n+1-k,l})=0 \quad (j, l=1, \cdots, n).$$

There is also defined the notion of Σ-derivative; some basic properties of the Σ-integral and Σ-derivative are introduced without proof. There are misprints and not all necessary restrictions are made: e.g., it is not stated that the region of variation of the arguments is simply connected. *A. D. Myškis* (RŽMat **1955**, no. 5026).

Boyarskiĭ, B. V. Homeomorphic solutions of Beltrami systems. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 661–664. (Russian)

The system of Beltrami equations which expresses the conformality of a mapping with respect to a Riemann metric may be written in the complex form $w_{\bar z}=\mu w_z$. Here $\mu(z)$ and the unknown function w are complex-valued functions and $|\mu(z)|\leq q_0<1$. The author establishes the existence of a solution in the large for the case where μ vanishes outside a large circle (unessential condition). For the case of a Hölder continuous μ this has been done in a recent paper by Vekua [same Dokl. (N.S.) **100** (1955), 197–200; MR **16**, 1114]. This paper is an extension of Vekua's note and deals with the case of a bounded measurable μ. The solution constructed is continuous, possesses generalized L_2 derivatives and gives a homeomorphic mapping of the whole plane onto itself. The proof is independent of the theory of uniformization and is based on the Calderón-Zygmund inequality [Acta. Math. **88** (1952), 85–139; MR **14**, 637].

The author also sketches a proof of the following results. The solution w takes measurable sets into measurable sets, sets of measures zero into sets of measure zero, and continuous functions having generalized L_2 derivatives into functions having the same properties. Furthermore, the formal rules of computing derivatives in new variables are valid. The inverse mapping has the same properties. This implies, in particular, that every solution of the Beltrami system is an analytic function of the solution considered, and that the Jacobian of the mapping constructed is different from zero almost everywhere. The author's construction also implies that the derivatives of the solution w are absolutely integrable to some power $p>2$.

As the author himself remarks, all these results, except the very last, have been obtained several years ago by Morrey [Trans. Amer. Math. Soc. **43** (1938), 126–166]. The author's proofs, however, are quite different.
L. Bers (New York, N.Y.).

Boyarskiĭ, B. V. On solutions of a linear elliptic system of differential equations in the plane. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 871–874. (Russian)

A uniformly elliptic linear system of partial differential equations of the form $v_y=\alpha u_x+\beta u_y$, $-v_x=\gamma u_x+\delta u_y$, can be transformed, by introducing new independent variables, into the canonical form (1) $w_{\bar z}=q(z)\overline{w_z}$. Here the unknown function $w=u+iv$ and the coefficient q are complex valued and $|q|\leq q_0<1$. The author considers this equation in a bounded domain D assuming that the coefficient q is a bounded measurable function. A solution is required to be continuous and to have generalized L_2 derivatives.

Theorem 1. Every solution of (1) may be written in the form (2) $w(z)=f[\chi(z)]$ where f is analytic and χ is a homeomorphism of the whole plane onto itself satisfying together with its inverse a uniform Hölder condition. The proof is based on the existence theorem for a Beltrami system [cf. the preceding review].

Theorem 2. Let D be the unit disc and assume that q satisfies a Hölder condition at the origin. Let $f(\chi)$ be an analytic function defined in the unit disc. Then there exists a homeomorphism $\chi(z)$ of the unit disc onto itself which keeps the origin fixed and is such that the function (2) is a solution of (1). The proof is based on the nonlinear differential equation (3) $\chi_{\bar z}=q(z)[\overline{f'(\chi)}/f'(\chi)]\overline{\chi_z}$ which must be satisfied by the function χ. A solution is sought in the form $\chi(z)=e^{\phi}$, where ϕ is required to be imaginary on the unit circle. The author transforms the differential equation (3) into an integral equation for the function $\phi_{\bar z}$. This integral equation is solved by means of the Schauder fixed-point theorem, the necessary a priori estimate being a consequence of the Calderón-Zygmund inequality. The author states that the condition on q at 0 could be omitted.

The author quotes the reviewer's result on the interiority of pseudoanalytic functions [Theory of pseudoanalytic functions, Inst. Math. Mech., New York Univ., 1953; MR **15**, 211], but believes that Theorem 1 in the present general formulation is new. Actually, this theorem is contained in the work of Morrey [Trans. Amer. Math. Soc. **43** (1938), 126–166]. The author also observes that Theorem 2 implies the Riemann mapping theorem for linear elliptic systems. Such a theorem has been proved by Z. Schapiro [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **30**, (1941), 690–692; MR **2**, 359]. The reviewer observes that, conversely, Schapiro's theorem in the form given in Bers and Nirenberg [Convegno internazionale sulle equazioni lineari alle derivate parziali, Trieste, 1954, pp. 111–140, Cremonese, Rome, 1955] can be used to derive Theorem 2. The paper just cited also contains an extension of Theorem 1 to systems involving terms of orders zero.
L. Bers (New York, N.Y.).

★ **Grinberg, G. A. On the solution of the equations of mathematical physics with partially or completely separable variables.** Sbornik posvyaščennyi semidesyatiletiyu akademika A. F. Ioffe [Collection in honor of the seventieth birthday of academician A. F. Ioffe], pp. 50–60. Izdat. Akad. Nauk SSSR, Moscow, 1950.

Let a function u of the variables $s_1, \cdots, s_m$ satisfy a linear partial differential equation of the form

$$(1) \qquad D[u]=f(s_1, \cdots, s_m),$$

where $D[u]=D_0(s_1)u+D_1(s_2, \cdots, s_m)u$ and

$$r(s_1)D_0(s_1)u=\frac{\partial}{\partial s_1}\left[p(s_1)\frac{\partial u}{\partial s_1}\right]+q(s_1)u,$$

together with inhomogeneous boundary conditions of Dirichlet, Neumann, or mixed type at the ends of the interval (a_1, b_1) over which s_1 ranges. Then u can be expanded in terms of the eigenfunctions Q_n of the equation $D_0(s_1)Q_n+\lambda_n Q_n=0$, satisfying homogeneous boundary conditions of the same type. The coefficients in the expansion satisfy a differential equation with fewer independent variables by one than (1). The present paper extends this method to the case where p, q, and r are piecewise continuous within (a_1, b_1). The results are used to solve the problem of heat conduction in a bar consisting of two homogeneous sections of different material.

In a second part the author uses Green's theorem to obtain a formula for the solution of the boundary-value problem in which $D[u]$ is the general self-adjoint expression in n independent variables, and an inhomogeneous boundary condition is imposed upon a closed surface containing an n-dimensional volume.

R. N. Goss (San Diego, Calif.).

★ **Grinberg, G. A. On the solution of the equations of mathematical physics with partially, or completely, separable variables.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 15 pp. (mimeographed) $15.00.

Translation of the paper reviewed above.

Levitan, B. M. On expansion in eigenfunctions of the Laplace operator. Mat. Sb. N.S. **35**(77) (1954), 267–316. (Russian)

This paper contains the proofs of the results announced by the author in Dokl. Akad. Nauk SSSR (N.S.) **90** (1953) 133–135; MR **15**, 129.

E. A. Coddington (Copenhagen).

Have

★ **Gårding, Lars. Eigenfunction expansions connected with elliptic differential operators.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 44–55 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

This paper deals with the extension to elliptic partial differential operators of the results on eigenfunction expansions for singular Sturm-Liouville ordinary differential operators originated by H. Weyl, and extended by Stone, Titchmarsh, Kodaira, M. Krein, and others. Results on the Schroedinger equation had previously been obtained by Carleman [Ark. Mat. Astr. Fys. **24B** (1934), no. 11] and extended by Povsner [Mat. Sb. N.S. **32** (74) (1953), 109–156; MR **14**, 755]. The results of the present paper are similar to those obtained independently by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 454–459, 459–463; MR **16**, 134] and are based upon general ideas similar to those put forward by F. I. Mautner [ibid. **39** (1953), 49–53; MR **14**, 659].

Let S be an open set in Euclidean n-space, L a formally self-adjoint linear elliptic differential operator of order m with coefficients defined and differentiable of a high order in the interior of S ($\bar{L}$ is obtained from L by taking the complex conjugates of the coefficients). Considered as an operator in $L^2(S)$ with domain $C_0^\infty(S)$, the m-times continuously differentiable functions with compact support in S, L is a symmetric operator. It is assumed that L has a self-adjoint extension Λ. (Such an extension will certainly exist if L has real coefficients and hence is a real operator.) Let $\{E_\lambda\}$ be the spectral resolution of Λ. For α, a bounded Borel measurable function with compact support on the real line R, set $(\alpha(\Lambda)f, g)=\int_R \alpha(\lambda)d(E_\lambda f, g)$.

The author shows that there exists a kernel $\theta(x, y, \alpha)$ in $C^\infty(S\times S)$ defined for each α above, linear in α, satisfying the equations $\bar{L}_x\theta(x, y, \alpha)=L_y\theta(x, y, \alpha)=\theta(x, y, \lambda\alpha)$, for which for all $f, g \in C_0(S)$, one has

$$(\alpha(\Lambda)f, g)=\int_S\int_S \theta(x, y, \alpha)f(x)\bar{g}(y)\,dxdy.$$

It is shown that there exists a measure $dt(\lambda)$ on R for which

$$\theta(x, y, \alpha)=\int\theta(x, y, \lambda)\alpha(\lambda)dt(\lambda),$$

with $\theta(x, y, \lambda)$ integrable on compact subsets of $S\times S\times R$ with respect to $dxdydt$. It may be arranged that, for almost all λ, $\theta(x, y, \lambda)$ belongs to $C^\infty(S\times S)$ and satisfies

$$\bar{L}_x\theta(x, y, \lambda)=L_y\theta(x, y, \lambda)=\lambda\theta(x, y, \lambda).$$

The final result, which is the main theorem on eigenfunction expansions proper, is the following: There exists a sequence of eigenfunctions $\{\theta_j(x, \lambda)\}$ for almost all λ, measurable with respect to $dxdt$ and satisfying $(L-\lambda)\theta_j(x, \lambda)=0$, for which

$$\theta(x, y, \lambda)=\sum_j \bar{\theta}_j(x, \lambda)\theta_j(y, \lambda).$$

Let H^* be the Hilbert space of vector-valued functions $F(\lambda)=\{F_j(\lambda)\}$ with inner product

$$[F, G]=\int\sum F_j(\lambda)\bar{G}_j(\lambda)dt(\lambda).$$

For $f \in C_0(S)$, let Zf be the element of H defined by $(Zf)_j(\lambda)=\int_S \bar{\theta}_j(x, \lambda)f(x)dx$. Then Z defines a unitary mapping of $L^2(S)$ upon H^* whose inverse is given by

$$(Z^{-1}F)(x)=\int\sum_j F_j(\lambda)\theta_j(x,\lambda)dt(\lambda),$$

for F with compact support in R. *F. E. Browder.*

Gårding, Lars. On the asymptotic properties of the spectral function belonging to a self-adjoint semi-bounded extension of an elliptic differential operator. Kungl. Fysiog. Sällsk. i Lund Förh. **24** (1954), no. 21, 18 pp. (1955).

Consider a linear differential operator

$$a=a(x, D)=\sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

where $D^\alpha=i^{-|\alpha|}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and the coefficients are sufficiently differentiable functions defined on an open set S in R^n. It is assumed that the polynomial $\hat{a}(x, \xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}\neq 0$ for $x \in S$ and $\xi=(\xi_1, \cdots, \xi_n)\neq 0$. Further let $C^k(S)$ be the set of k-times differentiable complex-valued functions on S, let $C_0^k(S)$ be the subset of functions with compact support, and let $H=H(S)$ be the Hilbert space $L_2(S)$. It is assumed that a, considered as an operator on $C_0^m(S)$ to $H(S)$, has a self-adjoint extension A which is bounded from below with lower bound $\mu(A)$. This requires that $\hat{a}(x, \xi)$ is a positive definite polynomial and that m is even. Finally introduce the operator $\hat{a}_z(D)=\hat{a}(z, D)=\sum_{|\alpha|=m} a_\alpha(z)D_z^\alpha$, where z is a fixed point in S. Considered as an operator on $C_0^m(S)$ to $H(R^n)$, it has a self-adjoint closure $\hat{A}_z$.

Let $E(\lambda)$ be the set of projections in the spectral resolutions of A so that $A=\int\lambda dE(\lambda)$. The author proves that for every λ there exists a Carleman kernel $e(\lambda, x, y)$ such that $[E(\lambda)f](y)=\int e(\lambda, x, y)f(x)dx$ for almost all y

when $f \in H$. Here the kernel is a Borel function on $R\times S\times S$, continuous on $S\times S$ for fixed λ and zero for $\lambda<\mu(A)$. Further, it is hermitian and its total variation over finite λ-intervals is bounded on compact subsets of $S\times S$. When applied to the operator $\hat{A}_s$, this theorem gives a similar representation for $[E_s(\lambda)f](y)$ by a kernel $\hat{e}_s(\lambda; x-y)$ where

$$\hat{e}_s(\lambda; u)=(2\pi)^{-n}\int_\Delta e^{-iu\xi}d\xi,\ u\xi=\sum_1^n u_k\xi_k,$$

and $\Delta=[\xi; \hat{a}(x, \xi)<\lambda]$. Here $\hat{e}_s(\lambda; u)=O(\lambda(n-1)/m)$ for $u\neq 0$. The author's fundamental theorem asserts that $e(\lambda, x, y)=\hat{e}_x(\lambda; x-y)+o(\lambda^{n/m})$ as $\lambda\to\infty$, uniformly on compact subsets of $S\times S$. This may be regarded as a unification of various special results, due to Carleman, Å. Pleijel, Minakshisundaram, Browder, and the author, concerning the asymptotic behavior of eigenfunctions of vibration problems. If a has constant coefficients, a sharper result may be proved. Let A_0 be the self-adjoint closure of a, considered as an operator on $C_0^m(R^n)$ to $H(R^n)$, let $\sigma(\lambda)$ be a function of bounded variation and set

$$I^k\sigma(\lambda)=(\lambda-\lambda_0)^{1-k}[\Gamma(k)]^{-1}\int_{\lambda_0}^{\lambda}(\lambda-\mu)^{k-1}d\sigma(\mu).$$

Then if $n>1$, $k\geqq 1$, $\lambda_0<\min[\mu(A), \mu(A_0)]$, one has

$$I^ke(\lambda, x, y)=I^ke_0(\lambda, x-y)+O[\lambda^{(n-k)/m}],$$

uniformly on compact subsets of $S\times S$. Here

$$e_0(\lambda; u)=(2\pi)^{-n}\int e^{-iu\xi}d\xi,$$

the integral taken over $\lambda>a(\xi)\equiv\sum_{|\alpha|\leq m} a_\alpha\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, is the spectral function of A_0. If $k=1$, one can replace e_0 by its "principal part" $\hat{e}_0$, but for $k>1$ this is impossible in general. For $k\geqq n-1$ this is an improvement of a result of B. M. Levitan [see the paper reviewed second above]. The proofs of the first two theorems are implicitly contained in earlier papers of the author [see the paper reviewed above]. The basic tool is a classical Tauberian theorem for a Stieltjes transform. In the case of the third theorem it is shown that the Laplace-Stieltjes transform of

$$e(\lambda, x, y)-e_0(\lambda, x-y)$$

is $O[\exp(-ct^{-1/(m-1)})]$ uniformly on compact subsets of $S\times S$ when $t<1$. The desired result then follows from a Tauberian theorem proved by T. Ganelius [same Förh. **24** (1954), no. 20; MR **17**, 147]. Various generalizations are indicated. *E. Hille* (New Haven, Conn.).

Have

★**Gårding, Lars. Applications of the theory of direct integrals of Hilbert spaces to some integral and differential operators.** The Institute for Fluid Dynamics and Applied Mathematics, Lecture series no. 11. University of Maryland, College Park, Md., 1954. 23 pp.

Si A est un opérateur self-adjoint dans un espace de Hilbert H on sait [théorème de J. von Neumann, Ann. of Math. (2) **50** (1949), 401–485; MR **10**, 548] qu'il existe un opérateur unitaire T de H sur H^* [H^* étant une somme mesurable d'espaces de Hilbert, construite sur la droite R $(\lambda\in R)$], qui diagonalise A, i.e. $TAT^*=\{\lambda\}$, $\{\lambda\}$ désignant l'opérateur de multiplication par λ. L'A. décrit de la façon suivante la somme H^*: on donne sur R une mesure $\geqq 0$, $d\sigma$; une fonction $\nu(\lambda)$ mesurable $d\sigma$, à valeurs dans les entiers $\geqq 0$, $+\infty$ compris; H^* est alors l'espace des (classes de) fonctions $\lambda\to F(\lambda)=(F_1(\lambda), F_2(\lambda), \cdots)$, à valeurs dans $C^{\nu(\lambda)}$, telles que

$$(F, F)=\int_R\sum_{k=1}^{\nu(\lambda)}|F_k(\lambda)|^2\,d\sigma(\lambda)<\infty.$$

Ceci rappelé, un problème naturel est de décrire de façon plus explicite la „transformation de Fourier" T, moyennant des hypothèses supplémentaires sur A. Le résultat fondamental est le suivant: on prend $H=L^2(X, d\alpha)$, espace des (classes de) fonctions de carré sommable sur X pour la mesure positive $d\alpha$ $(x\in X)$. Alors on a le Théorème 1. On suppose qu'il existe une fonction ϕ, mesurable $d\sigma$, presque partout non nulle, telle que l'opérateur $\phi(A)$ soit un opérateur de Carleman. Dans ces conditions, sur un ensemble dense D_0 de H, on a:

$$\text{(a)}\qquad (Tf)(\lambda)=((Tf)_k(\lambda)),\ (Tf)_k(\lambda)=\int_X\overline{E_k(\lambda, x)}f(x)d\alpha(x),$$

$E_k(\lambda, x)$ fonction définie pour $\nu(\lambda)\geqq k$, $x\in X$, $d\sigma\otimes d\alpha$ mesurable. Sur un ensemble dense de H^*, on a:

$$\text{(b)}\qquad (T^*F)(x)=\int_R\Big(\sum_1^{\nu(\lambda)}F_k(\lambda)E_k(\lambda, x)\Big)d\sigma(\lambda).$$

Les fonctions $x\to E_k(\lambda, x)$ sont des fonctions propres (dans un sens généralisé, i.e. non forcément éléments de H) de A, relativement à λ. Les formules (a) et (b) donnent donc le développement des éléments de H au moyen des fonctions propres de A. Le Théorème 1 généralise (avec une démonstration différente) un théorème de Mautner [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 49–53 (Mautner prenait $\phi(\lambda)=(\lambda+c)^{-1}$, $\Im c\neq 0$); MR **14**, 659].

L'A. applique ensuite le Théorème 1 à la situation suivante: sur un ouvert S de R^n, on donne un opérateur différentiel $L=r(x)^{-1}\sum_{|\alpha|\leq m}L_\alpha(x)D^\alpha$, $r(x)>0$, r et L_α étant indéfiniment (ou suffisamment) dérivables dans S. On suppose que L est elliptique dans S, et symétrique sur $C_0^m(S)$=espace des fonctions n fois continûment différentiables sur S à support compact dans S, i.e. $(Lf, g)=(f, Lg)$, le produit scalaire étant pris dans l'espace $H=L^2(S, r(x)dx)$. Soit A un prolongement self-adjoint de L (supposé exister) (un tel prolongement définit un problème aux limites associé à L). Théorème 2. Pour $k>m/2n$, l'opérateur $(A+i)^{-k}$ est un opérateur de Carleman [la démonstration de ce résultat s'appuie essentiellement sur les résultats de John, Proc. Symposium Spectral Theory and Differential Problems, Oklahoma Agric. and Mech. Coll., Stillwater, Okla., 1951, pp. 113–175; MR **13**, 349]. Grace au théorème 2, on peut appliquer le Théorème 1, et développer les fonctions de H au moyen des fonctions propres de l'opérateur différentiel A pour des conditions aux limites (auto-adjointes) données. On a ainsi une généralisation au cas des opérateurs différentiels sur $S\subset R^n$ des résultats de Weyl, Stone, etc., établis pour les opérateurs de Sturm-Liouville sur un intervalle de R. Ces développements généralisent ceux donnés par l'auteur [voir la deuxième œuvre analysée ci-dessus], si $r(x)=1$, par une méthode différente. [Cf. aussi, toujours si $r=1$, Browder, Bull. Amer. Math. Soc. **60** (1954), 64.]

J. L. Lions (Nancy).

Conte, S. D., and Sangren, W. C. An expansion theorem for a pair of singular first order equations. Canad. J. Math. **6** (1954), 554–560.

Let H be the Hilbert space of all functions

$$f(x)=(f_1(x), f_2(x))$$

having square integrable real components defined when $x\geqq 0$, with the scalar product $(f, g)=\int_0^\infty f(x)\cdot g(x)dx$, where $f\cdot g=f_1g_1+f_2g_2$. The elliptic differential operator L defined by $(Lf)_1=-f_2'+q_1f_1$, $(Lf)_2=f_1'+q_2f_2$, where q_1 and q_2 are real and integrable functions, is self-adjoint if its domain of definition is taken to be all absolutely continuous functions in H such that $Lf\in H$ and

$f_1(0)\cos\alpha+f_2(0)\sin\alpha=0$ for some fixed real α. The authors determine the associated generalized Fourier expansion. (Reviewer's remark. The existence of such an expansion for any self-adjoint extension of an elliptic differential operator with sufficiently smooth coefficients has been proved by the reviewer in the papers reviewed above and third above and by F. Browder [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 454–459, 459–463; MR **16**, 134].) Let $u=u(x,\lambda)$ be the solution of $Lu=\lambda u$ with real λ such that $u_1(0,\lambda)=-\sin\alpha$ and $u_2(0,\lambda)=\cos\alpha$ and let $f\to F=Tf$ be the generalized Fourier transform. Then $F(\lambda)=\int u(x,\lambda)\cdot f(x)dx$ and Parseval's formula reads $(f,f)=\int_{-\infty}^{+\infty}F(\lambda)^2\varrho(\lambda)d\lambda$, where ϱ is continuous and positive. In particular, the spectrum of L is simple and continuous and extends over the entire real axis. *L. Gårding.*

Simoda, Seturo. Sur le problème de Dirichlet discontinu dans l'équation $\Delta u=F(x, u, \text{grad } u)$. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **1954**, no. 3, 29–36.

We use the terminology and notation of the review of an earlier paper [Osaka Math. J. **6** (1954), 243–268; MR **16**, 706]; in that review, near the end, replace b^* by d^*. If f is defined on a subset e of $\bar{d}$ and $y\in\bar{e}$, put $f^*(y)=\limsup f(x)$ as $x\to y$, $x\in e$; similarly, $f_*(y)=\liminf f(x)$. Let $\underline{w}$ and $\overline{w}$ be bounded smooth functions such that $\underline{w}\in\underline{Z}$, $\overline{w}\in\overline{Z}$, and $\underline{w}\leq\overline{w}$ in d. Let h be a bounded function on d^* such that $\underline{w}_*\leq h_*\leq h^*\leq\overline{w}^*$ on d^* (text has $\underline{w}^*\leq h_*$). Let $F(x,v,q)$ be continuous and bounded on a suitable region and suppose that $\Delta\overline{w}(x)\leq F(x,\overline{w}(x),\nabla\overline{w}(x))$ and $\Delta\underline{w}(x)\geq F(x,\underline{w}(x),\nabla\underline{w}(x))$. Then there exists at least one $u\in\overline{Z}\cap\underline{Z}$ such that $\Delta u(x)=F(x,u(x),\nabla u(x))$ and $\underline{w}\leq u\leq\overline{w}$ in d while $h_*\leq u_*\leq u^*\leq h^*$ on d^*. *F. A. Ficken.*

Karp, Samuel N. An application of Sturm-Liouville theory to a class of two-part boundary-value problems. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-13** (1955). i+22 pp.

This boundary-value problem in partial differential equations arises in connection with the theory of propagation of electromagnetic waves in a waveguide: $u_{xx}+u_{yy}+k^2(y)u=0$ $(-\infty<x<\infty,\ 0<y<a)$, $u(x,0)=0$ $(-\infty<x<\infty)$, $u(x,a)=0$ $(-\infty<x<0)$, $u_y(x,a)=0$ $(0<x<\infty)$. The prescribed function $k(y)$ is positive. A formula for the unknown function $u(x,y)$ is obtained here in the form of a line integral in the complex plane, where the path of integration extends over a distorted form of the real axis. The integrand of the line integral involves a fundamental solution of the equation

$$\phi''(-\nu^2,y)+[-\nu^2+k^2(y)]\phi(-\nu^2,y)=0,$$

where ν is the complex variable of integration. It also involves factors $\exp(i\nu x)$ and $A(\nu)$, where the function $A(\nu)$ is described in terms of infinite products that involve the characteristic numbers of two associated Sturm-Liouville boundary-value problems in each of which the governing differential equation is the above equation in $\phi(-\nu^2,y)$. The author uses Sturm-Liouville theory, line integrals and some physical reasons to arrive at the formula. *R. V. Churchill* (Ann Arbor, Mich.).

★**Ladyženskaya, O. A. Smešannaya zadača dlya giperboličeskogo uravneniya. [The mixed problem for a hyperbolic equation.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 279 pp. 8.65 rubles.

Notations: $x=(X,t)$, $X=(x_1,\cdots,x_n)\in R^n$, $t(=x_0)=$ temps; $\Omega=$ouvert de R^n; $W^k(\Omega)=$espace des classes de fonctions $f\in L^2(\Omega)$ ainsi que toutes leurs dérivées (faibles) d'ordre $\leq k$, muni de sa structure naturelle d'espace de Hilbert (ce n'est pas exactement la définition de l'A. mais il y a identité des définitions lorsque Ω est un ouvert borné de frontière assez régulière, ce qui sera toujours le cas). Définition analogue sur un ouvert Q de $R_x^n\times R_t$.

Si D est un opérateur différentiel hyperbolique du deuxième ordre, les problèmes mixtes étudiés sont (cf. aussi point D du résumé) du type suivant: on donne $Q=\Omega\times\{0<t\}$, ou $Q_l=\Omega\times\{0<t<l\}$. On cherche u dans Q ou Q_l solution de $Du=f$, avec: (a) $u(X,0)=\varphi(X)$, $u_t(X,0)=\psi(X)$ (conditions initiales); (b) des conditions aux limites, du type Dirichlet: $u(X,t)=0$ sur $S=\partial\Omega$, t quelconque, ou bien du type Neumann (dérivée „normale" nulle sur S) etc. Les conditions aux limites peuvent être réalisées soit au sens usuel (en tout point), soit „en moyenne"; on dit selon les cas que u est solution usuelle ou solution presque partout (pp). Si les conditions initiales sont en outre réalisées en un sens „faible" (au sens des distributions) on dit que u est solution généralisée.

Ces problèmes sont étudiés par trois méthodes (A, B, C) et pour les conditions de Dirichlet, dans des ouverts non cylindriques, par une quatrième méthode (D).

A. Méthode de Fourier (Chap. II).

$$D=\frac{\partial^2}{\partial t^2}-L,\quad L=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(X)\frac{\partial}{\partial x_j}\right)-a(X),$$

où les a_{ij} sont réelles, L elliptique, $a(X)\geq 0$ (ce qui est d'ailleurs inutile). Les fonctions a_{ij} sont suffisamment dérivables (les hypothèses exactes variant avec les énoncés). Démonstration directe de l'unicité de la solution généralisée. Pour l'existence, on utilise, c'est classique, les fonctions propres de L relativement aux conditions aux limites du problème; on suppose d'abord $f=0$; on obtient ainsi la solution u, plus ou moins dérivable en t (à valeurs dans un espace du type $W(\Omega)$), selon les données initiales. [Rev.: ceci n'utilise pas le fait que L est d'ordre 2; pour les conditions de Dirichlet, il est en outre inutile que la frontière de Ω soit régulière.] L'A. étudie ensuite la régularité en X de u. Ceci repose sur des inégalités relatives à L (Chap. II, § 3); cf. aussi pour ce point Guseva, résumé ci après.

B. Méthode des différences finies (Chap. III). On a $(t=x_0)$:

$$D=\sum_{i,j=0}^{n}a_{ij}(X,x_0)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}a_i(X,x_0)\frac{\partial}{\partial x_i}+a(X,x_0),$$

avec $a_{00}=1$, $a_{ij}=a_{ji}$, $-\sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j\geq\alpha|\xi|^2$, $\alpha>0$; conditions aux limites de Dirichlet; démonstration directe de l'unicité de la solution généralisée. Pour l'existence: on suppose d'abord $a_{0i}=0$; on remplace D (noté ici L par l'A.) par un opérateur D_h aux différences finies; soit u_h la solution du problème correspondant. On établit des majorations pour u_h (cf. (25), p. 145). On prolonge les u_h à Q, et il est possible de la faire de façon que les fonctions prolongées demeurent dans un ensemble borné d'un espace de Hilbert convenable, d'où l'existence de la solution par extraction d'une suite faiblement convergente. On a ainsi existence et unicité d'une solution généralisée, avec en outre des majorations de la solution (qui signifient que la solution dépend continument des données). La méthode permet d'étudier la différentiabilité au sens usuel de la solution (§ 3 et 4). On peut également se débarrasser (§ 6) de l'hypothèse $a_{0i}=0$, en choisissant convenablement D_h. Court § (§ 5) pour le cas où Ω n'est pas borné [ce § a semblé peu précis au Rev.].

C. Méthode de la transformation de Laplace (Chap. IV et appendice II).

$$D=\frac{\partial^2}{\partial t^2}-L,\ L=\sum_{i,j=1}^{n} a_{ij}(X)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} a_i(X)\frac{\partial}{\partial x_i}+a(X).$$

L'A. suppose que f admet une transformée de Laplace en t et est régulière en X. La solution u peut alors être calculée par transformation de Laplace en t, cette transformation étant prise soit pour X fixé (Chap. IV) soit à valeurs dans un espace du type $W(\Omega)$ (Appendice II). Pour pouvoir prendre la transformée de Laplace à X fixé, l'A. établit d'intéressants résultats sur la régularité des solutions de $Lu-\lambda^2u=g$ (Théorème 11, p. 210). [Rev.: 1) il y a intéret à dégager la „fonction" de Green du problème mixte, qui est en fait une distribution à valeurs vectorielles [cf. Lions, Acta Math. **94** (1955), 13–153, Chap. II; pour les distributions à valeurs vectorielles, cf. les travaux de L. Schwartz et A. Grothendieck]; on en déduit la solution avec deuxième membre quelconque par composition à valeurs vectorielles; la distribution de Green peut toujours être calculée par transformation de Laplace en t. 2) Le fait que L soit du deuxième ordre n'intervient pas, non plus que le fait que Ω est borné (sauf peut être pour le théorème 11)].

D. Méthode d'approximation analytique (Chap. V). Développement d'une méthode de Krzyżański et Schauder [Studia Math. **6** (1936), 162–189; aussi Schauder, ibid., 190–198]. On considère cette fois un ouvert R de $R_x^n\times R_t$ d'un type particulier, mais non cylindrique et on donne les conditions aux limites de Dirichlet, sur les portions de frontière qui ont une orientation de temps. Dans le cas des coefficients et des données analytiques, on résout un problème de Cauchy, puis un problème de Goursat, dans $0\leqq t\leqq h$. On a alors la solution dans la partie de R comprise dans $0\leqq t\leqq h$, soit u. Mais $u(X,h)$, $u_t(X,h)$ ne sont plus analytiques. On les approche par des polynomes, pour lesquels on résout Cauchy, puis Goursat, dans $t\in[h,2h]$, et ainsi de suite, d'où une famille de solutions approchées; ces fonctions demeurent dans un ensemble borné d'un espace de Hilbert convenable (majorations des § 1 et 2). D'où l'existence d'une solution. Dans le cas où les coefficients ne sont pas analytiques, on les approche par des polynomes, et on utilise encore les majorations à priori, d'où l'existence dans le cas général. Grace à des majorations à priori sur les dérivées (§ 2), on peut également étudier la différentiabilité au sens usuel de u.

E. Cf. les résumés ci après de deux notes de l'A.

J. L. Lions (Nancy).

Guseva, O. V. On boundary problems for strongly elliptic systems. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1069–1072. (Russian)

Soit A un opérateur différentiel fortement elliptique d'ordre $2n$ dans un ouvert D de R^m, de frontière assez régulière. Soit $W^k(D)$ l'espace de Hilbert du résumé précédent (remplacer D par Ω). Soit $\mathring{W}^k(D)$ l'adhérence dans $W^k(D)$ des fonctions à support compact. L'opérateur A est un isomorphisme de $\mathring{W}^n(D)$ sur son dual. Si alors on donne f dans $W^l(D)$, $l\geqq 0$, la solution dans $\mathring{W}^n(D)$ de $Au=f$, a des propriétés supplémentaires. L'A. annonce ceci: si $f\in W^l(D)$, $l\geqq 0$, alors $u\in W^{2n+l}(D)$ (sous des hypothèses convenables sur les coefficient de A). Même chose pour les systèmes. L'A. annonce enfin que l'on a des résultats analogues pour les autres problèmes aux limites usuellement attachés à A: problème du type Neumann, Neumann-Dirichlet, etc. (P. 1070, l. 8, lire $\mathring{W}^n$ au lieu de $\mathring{W}^{2n}$.)

J. L. Lions (Nancy).

Ladyženskaya, O. A. On a method of proof of theorems on existence and uniqueness of solution of Cauchy's problem for hyperbolic equations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 17–20. (Russian)

$$L=\frac{\partial^2}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right)+a_0\frac{\partial}{\partial t}+\sum_{i=1}^{n} a_i(x,t)\frac{\partial}{\partial x_i}+a(x,t)$$

où $x\in R^n$, t=temps, $0\leqq t\leqq l$, les fonctions a_{ij} étant bornées sur tout compact ainsi qu'un nombre suffisant de leurs dérivées. On suppose que sur tout compact

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j\geqq\alpha|\xi|^2,\ \alpha>0.$$

On cherche u solution du problème de Cauchy: $Lu=f$, $u(x,0)=\varphi_0(x)$, $u_t(x,0)=\varphi_1(x)$, u étant localement (dans $t\geqq 0$) dans W^2, f, φ_0, φ_1 étant localement dans L^2, W^1, L^2 (notations des résumés précédents).

On considère Q, cone d'axe parallèle à l'axe des t, dont les surfaces latérales ont une orientation d'espace, l'angle de la normale extérieure avec l'axe des t étant aigu. Pour $u\in W^2(Q)$, on considère $Bu=(Lu, u(x,0), u_t(x,0))$, élément de $W=L^2(Q)\times W^1(\Omega)\times L^2(\Omega)$, Ω étant la base dans $t=0$; l'image de $W^2(Q)$ par B est dense dans W (lire W au lieu de T dans le lemme 1). Or par l'inégalité de l'énergie, cette image est fermée, et l'application est biunivoque (lire (7) au lieu de (9)). Donc B est un isomorphisme sur, ce qui résout le problème. Etude de la régularité de u, sous des hypothèses supplémentaires pour f, φ_0, φ_1.

Procédé d'approximation de la solution à l'aide de bases de $W^2(Q)$. [Cf., aussi, K. O. Friedrichs, Comm. Pure Appl. Math. **7** (1954), 345–392; MR **16**, 44.]

J. L. Lions (Nancy).

Ladyženskaya, O. A. On the solution of nonstationary operator equations of various types. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 207–210. (Russian)

Soit H un espace de Hilbert séparable. § 1: résolution de

$$Su=u_{tt}+S_1(t)u+S_2(t)u=f,\ u(0)=\varphi_0,$$

$t\to u(t)$ étant une fonction absolument continue de $t\geqq 0$ dans H; $S_1(t)$, $S_2(t)$ sont des opérateurs non continus, l'ensemble de définition de $S_1(t)$ étant indépendant de t, contenant celui de $S_2(t)$, $S_1(t)$ étant auto adjoint défini positif, et dans un certain sens, $S_2(t)$ étant plus petit que $S_1(t)$. Théorème d'existence et d'unicité, par une méthode analogue à celle du résumé précédent. § 2: problème analogue (avec deux données initiales) pour

$$S=\partial^2/\partial t^2+S_1(t)+S_2(t).$$

§ 3: problème analogue pour $S=\partial/\partial t-iS_1(t)$. *J. L. Lions.*

Zaidman, S. Sur les fonctions et les valeurs propres de l'équation $\Delta^{(m)}u+\lambda u=0$ pour le deuxième problème de frontière. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 79–85. (Romanian. Russian and French summaries)

Problème du type „Neumann" pour l'opérateur Δ^m, dans un ouvert Ω borné de R^n de frontière assez régulière, posé dans l'espace $W^m(\Omega)$ [cf. résumés précédents]; le système des fonctions propres est complet dans $L^2(\Omega)$.

J. L. Lions (Nancy).

Vasilache, S. Un nouveau problème aux limites pour les équations à dérivées partielles de type hyperbolique. Com. Acad. R. P. Romîne **1** (1951), 35–39. (Romanian. Russian and French summaries)

L'auteur pose et résoud (méthode de Riemann) pour

l'équation

$$\frac{\partial^2 z}{\partial x \partial y}+p(x, y)\frac{\partial z}{\partial x}+q(x, y)\frac{\partial z}{\partial y}+r(x, y)z=f(x, y)$$

le problème aux limites $z=h(y)$ pour $x=a$, $\partial z/\partial x=g(x)$ pour $y=b$. *Y. Fourès-Bruhat* (Princeton, N.J.).

Diaz, J. B., and Ludford, G. S. S. On the singular Cauchy problem for a generalization of the Euler-Poisson-Darboux equation in two space variables. Ann. Mat. Pura Appl. (4) **38** (1955), 33–50.

The authors deal with the solution of the singular Cauchy problem of the differential equation

$$(*)\ L_-(u)-hu\equiv u_{xx}+u_{yy}-u_{tt}-\frac{k}{t}u_t-hu=0,\ k>0,\ t>0;$$
$$u(x, y, 0)=g(x, y),\ u_t(x, y, 0)=0$$

($h(x, y, t)$ continuous for $t\geqq 0$ together with its derivatives h_x, h_y, h_{xx}, h_{yy}; $g(x, y)$ a twice continuously differentiable function if $k\geqq 1$, thrice continuously differentiable function if $0<k<1$). In § 1 the elementary solution, in the sense of Hadamard, is found for the Euler-Poisson-Darboux equation

$$\Delta u-u_{tt}-\frac{k}{t}u_t=0,\ \Delta=\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2},$$
$$u(x_1, \cdots, x_m, 0)=g(x_1, \cdots, x_m),\ u_t(x_1, \cdots, x_m, 0)=0$$

from the known fundamental solution of a corresponding elliptic equation, given by Diaz and Weinstein. This elementary solution (for $m=2$) is used to construct the solution of a regular Cauchy problem on $t=t_0>0$ for $L_-(y)=0$ and to construct the solution of a singular Cauchy problem on $t=0$ for the inhomogeneous equation $L_-(y)=f(x, y, t)$ (§§ 2, 4). The bounds for the solution and certain of its derivatives obtained in § 3 can be employed to show that some improper "Duhamel integrals" are nevertheless convergent. Finally in § 5 the given Cauchy problem (*) is solved by considering an equivalent problem in integral equations. Its equivalence has been demonstrated in § 4. *M. Pinl* (Cologne).

Mackie, A. G. Contour integral solutions of a class of differential equations. J. Rational Mech. Anal. **4** (1955), 733–750.

The following initial-value problems for the equation

$$(1)\qquad \frac{\partial^2 w}{\partial r \partial s}+\frac{N}{r+s}\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right)=0\ (N>0)$$

are solved in terms of contour integrals:

$$(a)\qquad \frac{\partial w}{\partial r}=-\frac{\partial w}{\partial s}=f(r)\ \text{on}\ r=s;$$

$$(b)\qquad \frac{\partial w}{\partial r}=\frac{\partial w}{\partial s}=g(r)\ \text{on}\ r=s;$$

$$(c)\qquad w=p(r)\ \text{on}\ s=s_1,\ w=q(s)\ \text{on}\ r=r_1.$$

Here f, g, p, q are given analytic functions, $f(-r)=f(r)$, $g(-r)=-g(r)$. The solutions of (a) and (b) are interpreted as solutions of Cauchy problems with initial data on $z=0$ for the equation of generalized axisymmetric potentials,

$$(2)\qquad \frac{\partial^2\varphi}{\partial z^2}+\frac{\partial^2\varphi}{\partial \varrho^2}+\frac{2N}{\varrho}\frac{\partial\varphi}{\partial\varrho}=0,$$

arising from (1) by the transformation $\varrho+iz=r$, $\varrho-iz=s$. Using contour integrals, the author also extends to arbitrary values of N the solution of the singular Cauchy problem for (2),

$$(d)\qquad \varphi=\text{given analytic function on}\ \varrho=0,$$

previously given by A. Weinstein for $N>0$ [Bull. Amer. Math. Soc. **59** (1953), 20–38; MR **14**, 749] and comments on the uniqueness of the solution [additional reference concerning uniqueness: Alfred Huber, Ann. of Math. (2) **60** (1954), 351–358; MR **16**, 258]. The well-known applications of (c) to one-dimensional motions of a perfect gas and of (d) to a generalized Tricomi equation are elaborated. *P. Henrici* (Washington, D.C.).

Trautman, A. On the proofs of "backward" uniqueness for some non-conservative fields describable by differential equations of the hyperbolic type. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 307–312.

The author proves a theorem that if a vector field I^ν in the four-dimensional continuum is such that from its equations an equation of continuity $I^\nu{}_{,\nu}+Q=0$ can be deduced, where $|Q|$ is bounded, then the vanishing of I^0 on a bounded region R of the three-dimensional continuum implies the vanishing of I^0 throughout the range of influence of R. From this he deduces the uniqueness of the solution of a hyperbolic system with initial conditions prescribed in R. This result includes as a consequence the establishment of "backward" uniqueness ($t<0$) for Maxwell's equations, with respect to which I^0 is the field energy density. *R. N. Goss* (San Diego, Calif.).

Corduneanu, C. L'approximation et la stabilité des solutions des équations hyperboliques. Com. Acad. R. P. Romîne **5** (1955), 21–26. (Romanian. Russian and French summaries)

L'auteur considère l'équation $Z_{xy}+AZ_x+BZ_y+CZ=D$ avec les conditions aux limites caractéristiques: $Z(x, 0)=f(x)$, $Z(0, y)=g(y)$ [$f(0)=g(0)$] (A, A_x, B, B_y, C, D, f' et g' continues dans un rectangle). Il met l'équation sous la forme $v_{xy}+av_x+cv=d$ avec $v(x, 0)=v(0, y)=0$ et construit une solution par des fonctions approximantes dont les différences en y satisfont à une relation différentielle en x [méthode de Rothe pour les équations paraboliques, Math. Ann. **102** (1930), 650–670]. Des formules de limitation donnent des théorèmes de stabilité. *Y. Fourès-Bruhat* (Princeton, N.J.).

Chuang, Feng-Kan. On the radiation problem of a quasi-linear parabolic differential equation in connection with a mathematical model of turbulence. Acta Math. Sinica **3** (1953), 316–327. (Chinese. English summary)

In this paper an equation relating to a mathematical model of turbulence is studied, namely the quasi-linear parabolic differential equation

$$(1)\qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}.$$

The main results are as follows. 1. A complete investigation of the similarity solutions of eq. (1) is made. If the distribution function of u is Gaussian, then the distribution function of $\partial u/\partial x$ is definitely skewed. 2. The equation (1) is then transformed to a linearized heat equation. The radiation problem is analyzed, and the solution is proved to be unique. 3. A particular example of the radiation problem is given and the conditions under which the non-linear solution approaches the corresponding linearized solution are studied. [The paper

is based on the author's thesis at the California Institute of Technology.] (Author's summary.)
C. C. Lin (Cambridge, Mass.).

See also: Kung Sun, p. 142; Chang, p. 145, Pignedoli, p. 168; Friedman, p. 178; Gagua, p. 194; Slobodyanskiĭ, p. 195.

Integral Equations, Equations in Infinitely Many Variables

Yoshimatsu, Senjiro. Sur les théorèmes des équations intégrales. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **1954**, no. 3, 1–5.

Following F. Riesz [Acta Math. **41** (1916), 71–98, p. 78] the author proves that if T is a linear continuous transformation on a linear normed complete space X to itself of the form $T=I-\lambda V$, where I is the identity transformation and V is completely continuous, then the characteristic values of V have no limiting point excepting infinity. Applied to the case where X is a Hilbert space and V is symmetric, it is easily shown that V has a characteristic value λ such that $\|V\|=|\lambda|$, and that $Vx=\sum_n \lambda_n^{-1}(x, x_n)x_n$, where $\{x_n\}$ form a complete orthonormal set of characteristic elements corresponding to the characteristic values λ_n. *T. H. Hildebrandt.*

Chang, Shih-Hsun. A relation between characteristic values and singular values of linear integral equations. Acta Math. Sinica **3** (1953), 200–207. (Chinese. English summary)

Let $K(x, y)$ be a L^2 kernel. Let $\{\mu_i\}$ be the characteristic values of the equation $\mu\int K(x, y)f(y)dy=f(x)$, so arranged that $|\mu_i|\leqq|\mu_{i+1}|$. Let $\{\lambda_i\}$ be the singular values of K arranged in increasing order. The inequality (1) $\lambda_1\leqq|\mu_1|$ is classical and already known to I. Schur [Math. Ann. **66** (1909), 488–510]. The more general inequality (2) $\lambda_1\lambda_2\cdots\lambda_n\leqq|\mu_1\mu_2\cdots\mu_n|$ is due to H. Weyl [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 408–411; MR **11**, 37]. The present paper contains a proof of (1), (2). Using his previous results [Trans. Amer. Math. Soc. **67** (1949), 351–367; Proc. London Math. Soc. (3) **2** (1952), 22–29; MR **11**, 523; **13**, 950], the author also proves the following result. For any L^2 kernel $K(x, y)$ and for any positive integer n, there exist $2n$ L^2 functions H_i $(1\leqq i\leqq 2n)$ of $2n$ variables such that the n-rowed determinant det $(K(s_i, t_j))$ is equal to

$$H_1*H_2*\cdots*H_{2n}(s_1, \cdots, s_n; t_1, \cdots, t_n),$$

where

$$H_1*H_2(s_1, \cdots, s_n; t_1, \cdots, t_n)$$

denotes the convolution

$$\int H_1(s_1, \cdots, s_n; u_1, \cdots, u_n)\cdot H_2(u_1, \cdots, u_n; t_1, \cdots, t_n)du_1\cdots du_n.$$

In case K has continuous partial derivatives with respect to x of order $\leqq p$, the number $2n$ of factors in this decomposition can be replaced by $2np$. According to the author, (1) and (2) were first proved by him before Weyl. However the reviewer is unable to find the statement (2) or its proof in author's abstracts [Bull. Amer. Math. Soc. **55** (1949), 871–873], where, according to the author, the first proof had appeared. *Ky Fan* (Notre Dame, Ind.).

Bădescu, Radu. Sur une équation fonctionnelle. Com. Acad. R. P. Române **2** (1952), 319–324. (Romanian. Russian and French summaries)

In the integral equation $\Phi(z)-\mu\int_a^b K(s)\Phi(z+s)ds=\psi(z)$, if $\psi(z)=\sum_{n=1}^\infty \psi_n e^{-\lambda_n z}$ with $\lambda_n\to\infty$ monotonically, then a formal expression for $\Phi(z)$ is

$$\sum_{n=1}^\infty \psi_n e^{-\lambda_n z}(1-\mu\int_a^b K(s)e^{-\lambda_n s}ds)^{-1}.$$

Assuming that the representation for $\psi(z)$ is uniformly convergent in a certain half-plane, the expression for $\Phi(z)$ defines a solution analytic in μ in the vicinity of the origin. Means for obtaining analytic continuations of $\Phi(z)$ as a function of μ beyond the successive characteristic values $\mu_n=[\int_a^b K(s)e^{-\lambda_n s}ds]^{-1}$ are discussed and it is indicated that the three fundamental theorems of Fredholm on integral equations hold. *T. H. Hildebrandt.*

Bădescu, Radu. Sur une équation fonctionnelle. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6** (1954), 789–795. (Romanian. Russian and French summaries)

With respect to the integral equation of the paper reviewed above, viz. $\Phi(z)-\mu\int_a^b K(s)\Phi(z+s)ds=\psi(z)$, C. Popovici [same Bul. **4** (1952), 527–531; MR **15**, 630] showed that for $\psi(z)=0$, the equation possesses solutions of the form $e^{-\lambda z}$ for a continuum of values of μ. The author points out that these solutions are not holomorphic in μ at $\mu=0$. There is also included a uniqueness proof for the special class of solutions considered, excepting the values of μ for which $1-\mu\int_a^b K(s)e^{-\lambda_n s}ds=0$. *T. H. Hildebrandt.*

Ghermănescu, M. Sur une équation intégrale de type Volterra. Com. Acad. R. P. Române **1** (1951), 151–155. (Romanian. Russian and French summaries)

In the Volterra integral equation

$$\varphi(x)+\int_0^x K(x-s)\varphi(s)ds=f(s)$$

it is assumed that

$$K(x-y)=\sum_{n=0}^\infty \frac{a_n(x-y)^n}{n!},\quad f(x)=\sum_{n=0}^\infty \frac{b_n x^n}{n!},\quad \varphi(x)=\sum_{n=0}^\infty \frac{c_n x^n}{n!},$$

where $\Psi(x)=\sum_n a_n x^n$, $F(x)=\sum_n b_n x^n$ and $\Phi(x)=\sum_n c_n x^n$ converge in some vicinity of the origin. Then the c's are determined by the equations

$$b_n-c_n=a_0c_{n-1}+a_1c_{n-2}+\cdots+a_{n-1}c_0$$

[observed earlier by G. Prasad, Proc. Edinburgh Math. Soc. **42** (1924), 46–59, p. 54]. The author notes that these relations lead to the identity $\Phi(x)(1+x\Phi(x))=F(x)$. φ is the Pincherle adjoint [Mem. Accad. Sci. Ist. Bologna (4) **9** (1888), 45–71 = Acta Math. **48** (1926), 279–304] or Borel transform of Φ and these functions are related to each other by the formulas

$$\varphi(x)=\frac{1}{2\pi}\int_C e^{xz}\Phi\left(\frac{1}{z}\right)\frac{dz}{z}\quad\text{and}\quad \Phi(z)=\int_0^\infty e^{-x}\varphi(xz)dx,$$

where C is a circle containing all the singularities of $\Phi(1/z)/z$. A similar procedure is valid for the integral equation of the first kind. *T. H. Hildebrandt.*

Gohberg, I. C. On systems of singular integral equations. Uč. Zap. Kišinevsk. Univ. **11** (1954), 55–60. (Russian)

The author considers the system of singular integral equations

$$(1)\quad \sum_{k=1}^n \left\{a_{jk}(t)\varphi_k(t)+\frac{b_{jk}(t)}{\pi i}\int_\Gamma \frac{\varphi_k(\tau)}{\tau-t}d\tau+T_{jk}\varphi_k\right\}=f_j(t),$$

where Γ is a contour formed of one or several smooth closed nonintersecting curves, $a_{jk}(t)$ and $b_{jk}(t)$ are continuous functions and T_{jk} are operators completely continuous in $L_2(\Gamma)$. There is introduced the space $L_2^{(n)}(\Gamma)$ whose elements are the n-tuples $\varphi=\{\varphi_1, \cdots, \varphi_n\}$ and with the scalar product defined by

$$(\varphi, \psi)=\sum_{k=1}^{n}\int_{\Gamma}\varphi_k(\tau)\bar{\psi}_k(\tau)|d\tau|.$$

The left side of (1) defines an operator D bounded in $L_2^{(n)}(\Gamma)$.

Let A and B be matrices with elements $a_{jk}(t)$ and $b_{jk}(t)$ respectively. It is known that if $\det(A+B)$ and $\det(A-B)$ vanish nowhere on Γ, then F. Noether's theorems hold: 1) each of the equations $Du=0$ and $D^*u=0$ has a finite number of linearly independent solutions; 2) both equations are normally solvable. It is proved that vanishing of these determinants is not only sufficient but also necessary for Noether's theorems to hold. The proof is based on I. M. Gel'fand's theory of normed rings. *S. G. Mihlin* (RŽMat **1955**, no. 5072).

Stanković, Bogoljub. Sur une classe d'équations intégrales singulières. Srpska Akad. Nauka. Zb. Rad. **43**. Mat. Inst. **4** (1955), 81–130. (Serbo-Croatian. French summary)

The solution of the integral equation

$$\int_0^\infty e^{-sx}K(x, t)dx=s^{\mu}e^{-ts}$$

is given, where

$$K(x, t)=K_{\mu,\nu}(x, t)=x^{-\mu-1}\Phi(-\mu, -\nu; -tx^{-\nu}),$$

and $\Phi(\beta, \varrho; z)$ is Wright's function which is a generalization of a Bessel function [E. M. Wright, J. London Math. Soc. **8** (1933), 71–88]. The functional transformation

$$P_\nu(\sigma, f)=\int_0^\infty F_\nu(\sigma/x^{1/\nu})f(x)\frac{dx}{\nu x}$$

is studied, where $F_\nu(\sigma/x^{1/\nu})=xK_{0,\nu}(x, t)$. An elegant derivation of the solution of the functional equation $\varphi(x)=\lambda\varphi(x^n)$ is given. The paper concludes with applications of the results to a problem in physics, and with the derivation of certain relations between the Laplace transformation and Wright's function mentioned above.
H. P. Thielman (Ames, Iowa).

Vasilache, Sergiu. Sur la répartition spectrale des valeurs du paramètre λ dans les systèmes d'équations intégro-différentielles linéaires du type Volterra. Com. Acad. R. P. Romåne **2** (1952), 311–318. (Romanian. Russian and French summaries)

Methods used in a preceding paper [Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 7–18; MR **15**, 630] are applied to a system of linear integro-differential equations, viz.,

$$\sum_{g=1}^{n}\sum_{i=0}^{m}H_{hgi}(x)\varphi_g^{(i)}(x)=\lambda\int_a^x\sum_{g=1}^{n}\sum_{i=0}^{m}K_{hgi}(x,s)\varphi_g^{(i)}(s)+f_h(x),$$

and similar results are obtained. *T. H. Hildebrandt.*

Vasilache, S. Sur la solution générale des équations intégro-différentielles linéaires, à limites fixes d'intégration. Com. Acad. R. P. Romåne **1** (1951), 1–6. (Romanian. Russian and French summaries)

The solution of the integro-differential equation

$$\sum_{i=0}^{n}H_i(x)y^{(i)}(x)=f(x)+\lambda\int_a^b\sum_{r=0}^{m}K_r(x, s)y^{(r)}(s)ds$$

is transferred to that of an equation of the form

$$y(x)=h(x)+\lambda\int_a^b\sum_{r=0}^{m}P_r(x, \Delta)y^{(r)}(s)ds$$

by using a fundamental system of solutions of the homogeneous linear differential equation

$$\sum_{i=0}^{n}H_i(x)y^{(i)}(x)=0.$$

[See Vasilache, Acad. Repub. Pop. Romåne. Bul. Şti. Ser. Mat. Fiz. Chim. **2** (1950), 495–501; MR **13**, 354.]
T. H. Hildebrandt (Ann Arbor, Mich.).

Vasilache, S. Une nouvelle méthode de résolution des équations intégro-différentielles linéaires à deux ou à plusieurs variables indépendantes. Com. Acad. R. P. Romåne **1** (1951), 399–402. (Romanian. Russian and French summaries)

In the integro-differential equation

$$\Delta u(M)=f(M)+\lambda\int_{D_p}K(M, P)\Delta' u(P')dP,$$

let $u(M)$ and $f(M)$ be functions on an n-dimensional region with $M=(x_1, \cdots, x_n)$, D_p a p-dimensional region, $p<n$, $P'=(s_1, \cdots, s_p, x_{p+1}, \cdots, x_n)$ if $P=(s_1, \cdots, s_p)$, and Δ and Δ' linear partial differential operators relative to M and P' respectively. Then the solution of such an equation is reducible to one of the form

$$u(M)=g(M)+\lambda\int_{D_p}R(M, P)\Delta' u(P')dP$$

if a solution of $\Delta u(M)=F(M)$ can be expressed in the form

$$u(M)=h(M)+\int_{D_n}G(M, Q)F(Q)dQ.$$

T. H. Hildebrandt (Ann Arbor, Mich.).

Vasilache, Sergiu. Sur quelques nouveaux problèmes aux limites pour certaines classes d'équations intégro-différentielles ou aux dérivées partielles. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **6** (1955), 55–78. (Romanian. Russian and French summaries)

The integro-differential equation considered is

$$\sum_{p=0}^{m}\sum_{q=0}^{n}H_{pq}(x, y)\phi_{x^py^q}^{(p+q)}(x, y)=f(x, y)+$$
$$\lambda\int_a^x\int_b^y\sum_{p=0}^{m}\sum_{q=0}^{n}K_{pq}(x, y; u, v)\,\phi_{u^pv^q}^{(p+q)}(u, v)dudv,$$

where a solution is sought subject to the initial conditions

$$\phi_{x^s}^{s}(x, y)|_{x=a}=g_s(y),\ s=0, 1, \cdots, n-1;$$
$$\phi_{x^ny^r}^{(n+r)}(x, y)|_{y=b}=h_r(x)\quad (r=0, 1, \cdots, m-1).$$

Following the method of a previous paper [Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 7–18; MR **15**, 630], the partial derivatives $\phi_{x^sy^r}^{s+r}(x, y)$ for $s+r<p+q$ are expressed in terms of $\phi_{x^my^n}^{m+n}(x, y)$, resulting in a Volterra mixed integral equation:

$$H(x, y)\Phi(x, y)+\int_a^xK(x, y, u)\Phi(u, y)du+$$
$$\int_b^yL(x, y, v)\Phi(x, v)dv+\lambda\int_a^x\int_b^yM(x, y, u, v)\Phi(u, v)dudv$$
$$=F(x, y)+\lambda G(x, y)$$

in $\phi_{x^my^n}^{m+n}(x, y)=\Phi(x, y)$, leading to theorems on the solution of the integro-differential equation.
T. H. Hildebrandt (Ann Arbor, Mich.).

Vasilache, Sergiu. Sur la solution de l'équation
$$\sum_1^m {}_i H_i(x)\varphi^{(i)}(x)=f(x)+\lambda\int_a^x\left(\frac{K(x,s)}{(x-a)^p}\right)\varphi^{(n)}(s)ds.$$
Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 87–95. (Romanian. Russian and French summaries)

In the integro-differential equation of the title it is assumed that $m<n$, $p>1$ and $K(x,x)\neq 0$ on (a,b). The well known identity
$$\varphi^{(i)}(x)=\varphi^{(i)}(a)+\varphi^{(i+1)}(a)(x-a)+\varphi^{(n-i-1)}(a)\frac{(x-a)^{n-i-1}}{(n-i-1)!}+$$
$$+\int_a^x\frac{(x-s)^{n-i-1}}{(n-i-1)!}\varphi^{(n)}(s)ds$$
is used to reduce the equation to an integral equation of the first kind in $\varphi^{(n)}(s)$. If $m<n-1$, the customary operation of taking the derivative of the equation results in an equation of the form
$$\varphi^{(n)}(x)=\mu(x-a)^{p-1}F(x)+\int_a^x[(x-a)^{p-1}Q_1(x,s)$$
$$+\mu Q_2(x,s)]\varphi^{(n)}(s)\,ds,$$
where $\mu=1/\lambda$, which has a solution in $\varphi^{(n)}(x)$ in a permanently convergent power series in μ and for which $\varphi^{(n)}(a)=\varphi^{(n+1)}(a)=\varphi^{(n+p-2)}(a)=0$. When $m=n-1$, the coefficient of $\varphi^{(n)}(x)$ involves the expression
$$\lambda K(x,x)-(x-a)^pH_{n-1}(x)$$
so that values of λ for which this expression vanishes on (a,b) are barred. *T. H. Hildebrandt.*

Haimovici, Ad. Sur un système d'équations intégro-différentielles. Com. Acad. R. P. Romåne 1 (1951), 13–18. (Romanian. Russian and French summaries)

L'auteur étudie un système d'équations intégro-différentielles, dont certaines sont linéaires, généralisant des équations de biologie mathématique dont la théorie a été commencée par Volterra [Leçons sur la théorie mathématique de la lutte pour la vie, Gauthier-Villars, Paris, 1931]. On donne des résultats d'existence et unicité des solutions, obtenus par approximation, successive et on montre comment on peut préciser les limites pour $t\to+\infty$, lorsqu'on suppose qu'il y a des limites fines. *M. Brelot.*

Magnus, W. An infinite system of linear equations arising in diffraction theory. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-80** (1955), i+19 pp.

In the problem of diffraction of plane sound waves incident normally on a soft plane screen with a circular aperture, the velocity potential u satisfies $\Delta u+K^2u=0$ and vanishes on the screen. If we use cylindrical coordinates so that $z=0$, $0\leqq\varrho\leqq a$, is the aperture, the value of u in the aperture can be expanded as a series of the form
$$\sum_{n=0}^{\infty}(-1)^n\frac{n!}{\Gamma(n+{}^3/_2)}s_nP_{2n+1}((1-\varrho^2a^{-2})^{\frac12}),$$
where the coefficients s_n satisfy a system of infinitely many linear equations
$$\sum_{n=0}^{\infty}g_{m,n}s_n=t_m\quad(m,n=0,1,\cdots),$$
where
$$g_{mn}=(-1)^{m+n}\int_0^{\infty}(v^2-1)^{\frac12}J_{2m+{}^3/_2}(\alpha v)J_{2n+{}^3/_2}(\alpha v)\frac{dv}{v^2},$$
α being equal to Ka, and $t_m=-i\alpha(3\pi^{\frac12})^{-1}\delta_{m,0}$, where $\delta_{m,n}=0$ $(m\neq n)$, $\delta_{n,n}=1$.

It is shown in this paper that the system of equations can be used effectively for the purpose of computing the field in the aperture and the transmission coefficient. This can be done approximately for all real values of α by showing the first $N+1$ equations for the first $N+1$ unknows $s_0, s_1, \cdots, s_N$, the remaining s_n being supposed zero.

General information on the behaviour of s_n, regarded as a function of α is also obtained: s_n is an analytic function of the complex variable α with no singularity on the imaginary axis. *E. T. Copson* (St. Andrews).

See also: Hammersley and Nelder, p. 170; Riesz and Sz.-Nagy, p. 175.

Theory of Probability

Mihoc, Gh. Une application de la méthode représentative. Com. Acad. R. P. Romåne 1 (1951), 1013–1016. (Romanian. Russian and French summaries)

Examination of a simple urn scheme. *J. L. Doob.*

Godwin, H. J. On generalizations of Tchebychef's inequality. J. Amer. Statist. Assoc. 50 (1955), 923–945.

Comprehensive survey giving 38 Tchebychef-type inequalities applying to single variate distributions, averages or totals, and multivariate distributions. *G. E. Noether* (Boston, Mass.).

Marinescu, G. Une généralisation de la notion de variable aléatoire. Com. Acad. R. P. Romåne 1 (1951), 45–50. (Romanian. Russian and French summaries)

Probability theory is based on a measure defined on a Boolean algebra, so that a random variable is the corresponding analogue of a point function (Ortsfunktion). This approach has been given in detail by Kappos [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948,** 309–320 (1949); MR **11,** 443]. *J. L. Doob* (Urbana, Ill.).

Bazu [Basu], D. Note on mappings of probability spaces. Vestnik Leningrad. Univ. **10** (1955), no. 5, 33–35. (Russian)

If ξ_1 and ξ_2 are random variables, not necessarily defined on the same measure space, the author writes $\xi_1\rightrightarrows\xi_2$ if, for some Borel measurable f, the random variables $f(\xi_1)$ and ξ_2 have the same distribution. He calls the original pair of random variables equivalent if the relationship holds in both directions. Then any two random variables with continuous distribution functions are equivalent. The natural extensions to relationships between families of random variables are formulated. To investigate such relationships the author maps any discrete stochastic process $\{X_n, n\geqq1\}$ on the process $\{Y_n, n\geqq1\}$ of independent random variables uniformly distributed on the interval $[0,1]$ by the standard map involving the successive distribution functions. [See Lévy, Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937.] *J. L. Doob* (Urbana, Ill.).

Smith, Walter L. Extensions of a renewal theorem. Proc. Cambridge Philos. Soc. **51** (1955), 629–638.

Given independent random variables $(x_n\colon n>0)$ with the common distribution, $\Pr(x_n\leqq x)=\int_{t\leqq x}f(t)dt$, such that

$0<\mu=E(x_n)\leq+\infty$ and $f(t)\in L_{1+\epsilon}$ for some $\epsilon>0$, set $f_1(x)=f(x)$, $f_{n+1}(x)=\int f(x-t)f_n(t)dt$, and $h(x)=\sum_{n>0}f_n(x)$, and consider the conditions

$$(-)\ f(x)=o(1)\ (x\downarrow-\infty) \text{ and } (+)\ f(x)=o(1)\ (x\uparrow+\infty).$$

Using a trick due to Blackwell [Pacific J. Math. **3** (1953), 315–320; MR **14**, 994], the author shows that $(-)\Rightarrow h(x)=o(1)\ (x\downarrow-\infty)$ and that $(+)\Rightarrow h(x)=\mu^{-1}+o(1)$ $(x\uparrow+\infty)$, the second statement being an improvement on an earlier result [Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954), 9–48; MR **15**, 722] in which the $(x_n:n>0)$ were supposed to be positive and the mean μ to be $<+\infty$.

Reviewer's note: In l. 24 on p. 632, the N_i should be replaced by N_i-N_{i-1} with the understanding that $N_0=0$.

H. P. McKean, Jr. (Princeton, N.J.)

Milicer Grużewska, Halina. L'arithmétique des variables aléatoires. Cahiers Rhodan. **6** (1954), 9–56.

Expository paper, concerned with the distributions of sums and products of independent random variables, and limit distributions of sequences of random variables.

J. L. Doob (Urbana, Ill.).

Dobrušin, R. L. Two limit theorems for the simplest random walk on a line. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 139–146. (Russian)

Let τ_n be the position of a particle after making n steps of a one-dimensional symmetric unit-step random walk. Let f be a function on the integers, with $\sum_{-\infty}^{\infty}f(j)=c$, where the series converges absolutely. Then, if $c\neq0$, the distribution of $\sum_1^n f(\tau_j)/(cn^{\frac{1}{2}})$ converges to a limit distribution. If $c=0$, and if f vanishes except on a finite set, the distribution of $\sum_1^n f(\tau_j)/(dn^{\frac{1}{4}})$ converges to a limit distribution. Let ξ, η be mutually independent Gaussian random variables with zero expectations and unit variances. Then the above limit distributions are those of $|\xi|$, $|\xi|^{\frac{1}{2}}\eta$ respectively, and d is a positive constant evaluated explicitly in terms of f. Feller [Trans. Amer. Math. Soc. **67** (1949), 98–119; MR **11**, 255] proved a special case of the first theorem. Related results have been proved by Kallianpur and Robbins [Duke Math. J. **21** (1954), 285–307; MR **16**, 52.]

J. L. Doob (Urbana, Ill.).

Ginzburg, G. M. On limit distributions determined by stochastic equations with an infinite set of zeros of the dispersion function. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 441–444. (Russian)

In a previous paper [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 563–580; MR **13**, 475] the author established necessary and sufficient conditions for the uniqueness of the limit distribution, as $t\to\infty$, determined by the stochastic equation $\Delta y=A(y)\Delta t+f(\alpha,y)(\Delta t)^{\frac{1}{2}}$, assuming that $A(y)$ and $B(y)=E\{f^2\}$ define analytic functions on the real axis, and that B has only finitely many zeros there. The result now is extended to allow B to have infinitely many zeros, and examples are given. *J. L. Doob.*

Machado, Emilio Antonio. On the random function $X(t)=\int_{t_0}^t K(t,s)d\zeta(s)$. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. Serie Tercera. Publ. Esp. **48** (1954), no. 211, 23 pp. (Spanish)

For the random process $X(t)=\int_{t_0}^t K(t,s)d\zeta(s)$, where $\zeta(s)$ is a process with independent increments such that the distribution of $\zeta(s+h)-\zeta(s)$ depends only on h, formulas are given for the characteristic function of $X(t_1)$, $\cdots$, $X(t_n)$ and the covariance function. If $K(t,s)=u(t)v(s)$, $X(t)$ is a Markov process. If K is a function of $t-s$ only and if $t_0=-\infty$, $X(t)$ is a stationary process.

D. Blackwell (Berkeley, Calif.).

Karlin, Samuel, and McGregor, James. Representation of a class of stochastic processes. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 387–391.

(1) Let P be a matrix $\{P_{ij}\}$ of non-negative elements vanishing save on the main and on its two flanking diagonals and having row-sums not exceeding unity. Then the elements of P^n will be the n-step transition probabilities in a random walk with the possibility of permanent loss to a universally accessible (ignored) state. Let $Q_i(\lambda)$ be the roots of the equation $\sum P_{i\alpha}Q_\alpha(\lambda)=\lambda Q_i(\lambda)$ $[Q_0(\lambda)\equiv1]$; these are polynomials and there is a unique positive regular measure ψ on $[-1,1]$, of total mass 1, such that the polynomials form an orthogonal system with regard to ψ and

$$(*)\qquad P_{ij}{}^n=\pi_j\int_{-1}^{1}\lambda^nQ_i(\lambda)Q_j(\lambda)\psi(d\lambda),$$

where $\pi_0\equiv1$ and $\pi_j\equiv p_0p_1\cdots p_{j-1}/(q_1q_2\cdots q_j)$ $(j\geq1)$. Here the p's are the super- and the q's are the sub-diagonal elements of P (it's supposed that they are all positive).

(2) The authors then set up a representation analogous to (*) for the transition probabilities in a general birth-and-death process (as before, with the possibility of loss from any state to an ignored absorbing state). If A is the matrix of coefficients in the usual 'backward' and 'forward' equations, the authors set up a one-to-one correspondence between the transition matrices $\{P(t)\}$ satisfying I: $P'(t)=P(t)A$; II: $P'(t)=AP(t)$; III: $P(0)=I$; IV: $P(t)\geq0$; V: $\sum_\alpha P_{i\alpha}(t)\leq1$; VI: $P(t+s)=P(t)P(s)$; and the extremal solutions to a Stieltjes moment problem which can be formulated in terms of A. The matrices $P(t)$ are given in terms of the extremal solutions ψ to the moment problem by the appropriate analogue (**) of (*). A minimal solution plays its customary role [Feller, Trans. Amer. Math. Soc. **48** (1940), 488–515; **58** (1945), 474; MR **2**, 101]. The authors give a necessary and sufficient condition for the minimal solution to be "honest"; in this they appear to have been anticipated by Dobrušin [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6(52), 185–191; MR **14**, 567]. Dobrušin's procedure was entirely different, and the author's approach has more in common with one of Ledermann and Reuter [Philos. Trans. Roy. Soc. London. Ser. A. **246** (1954), 321–369; MR **15**, 625] of which the authors were evidently unaware. Ledermann and Reuter's results overlap those of Karlin and McGregor, but there is no inclusion relation; the two papers should be studied together.

The authors state that the existence of the non-zero finite limit of $P_{ij}(t)/P_{kl}(t)$ (as $t\uparrow\infty$) follows easily from (**), and they also assert that if $0\leq i_1<i_2<\cdots<i_n$, $0\leq j_1<j_2<\cdots<j_n$, then $\det\{P_{i_p,j_q}(t)\}>0$ for $t>0$.

(3) It is claimed that diffusion processes can be discussed in a similar manner, and that the method can be extended to more than one "spatial" dimension.

No proofs are given, but a detailed account is forthcoming. It would be convenient if this were to incorporate also the Ledermann-Reuter results noted above.

D. G. Kendall (Oxford).

Maruyama, Gisirō. Continuous Markov processes and stochastic equations. Rend. Circ. Mat. Palermo (2) **4** (1955), 48–90.

Proofs of results announced in a previous paper [Nat. Sci. Rep. Ochanomizu Univ. **4** (1953) 40–43; MR **15**, 448].

J. L. Doob (Urbana, Ill.).

Neveu, Jacques. Jeux de Markoff et problèmes d'absorption. C. R. Acad. Sci. Paris **240** (1955), 2372–2374.

Let Φ_s be a semi-group defined by a Markoff process and let $x(t)$ be a random variable for t on $(0, \infty)$ subject to the transition probabilities of this process. For every Baire function f such that $f(x(t))$ has expectation for every t, let Hf be defined as the maximal Baire function satisfying $Hf \leq f$ and $Hf \leq \Phi_s Hf$ for all s; $Hf(x(t))$ is the maximal semi-martingale majorized by $f(x(t))$. Let

$$\Lambda_f = \{x; Hf(x) = f(x)\}$$

and let the random variable $Z(\omega)$ be defined to be equal to $f(x(t;\omega))$ if $S_f(\omega) = t$, S_f being defined by

$$\{\omega; S_f(\omega) > s_2\} = \{\omega; Hf(x(s_1;\omega)) < f(x(s_2;\omega)) \text{ for } s_1 \leq s_2\}.$$

The author states, without proof, that if the process is ergodic the function $G(x) = E\{g(Z)|f(x(0)) = x\}$ gives the solution of the Dirichlet problem $AG = 0$ with the boundary condition $G = g$ on Λ_f. Here A denotes the infinitesimal generator of the semi-group. *K. Yosida* (Tokyo).

Yosida, Kôsaku. On the generating parametrix of the stochastic processes. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 240–244.

Let $\mathfrak{X}$ be a locally compact space, let $\mathfrak{B}$ be the Borel field generated by the open subsets of $\mathfrak{X}$, and let $C(\mathfrak{X})$ be a certain (B)-space of continuous functions containing all such functions with compact carriers as well as $f(x) \equiv 1$. A function $G(x, E)$ satisfying the conditions $G(x, E) \geq 0$, $G(x, X) = 1$, $G(x, E)$ is measurable in x for each $E \in \mathfrak{B}$ and a measure in E for each $x \in \mathfrak{X}$, is called a transition probability. A Markoff process is a one-parameter family of transition probabilities $[P(t, x, E); t > 0]$ satisfying the further condition $P(t+s, x, E) = \int_{\mathfrak{X}} P(t, x, dy) P(s, y, E)$. If in addition $\lim_{t\to 0+} P(t, x, \mathfrak{X} - N(x)) = 0$ for each neighborhood $N(x)$ of x and if $(P_t f)(x) = \int_{\mathfrak{X}} f(y) P(t, x, dy)$ defines a linear operator on $C(\mathfrak{X})$ to itself, then $[P(t, x, E)]$ is said to belong to class $\mathfrak{P}$. Finally let $\mathfrak{G}$ be the class of all transition probabilities such that $(Gf)(x) = \int_{\mathfrak{X}} f(y) G(x, dy)$ defines a 1-1 map of $C(\mathfrak{X})$ into a dense subspace of itself and such that for any α, $0 < \alpha < 1$, the linear operator $\alpha I + (1-\alpha) G$ defines a 1-1 map of $C(\mathfrak{X})$ onto itself with $G_\alpha = G[\alpha I + (1-\alpha) G]^{-1}$ on $C(\mathfrak{X})$ into $C(\mathfrak{X})$ defined by a transition probability $G_\alpha(x, E)$. With these conventions the main result is the following: For any $[P(t, x, E)] \in \mathfrak{P}$, the transition probability $G(x, E) = \int_0^\infty \exp(-t) P(t, x, E) dt$ belongs to $\mathfrak{G}$. Conversely, any $G(x, E) \in \mathfrak{G}$ defines $[P(t, x, E)] \in \mathfrak{P}$ by the linear operator

$$P_t f = \text{strong} \lim_{\alpha\to 0} \exp\{t(G-I)[\alpha I + (1-\alpha) G]^{-1}\}$$

in such a way that $(P_t f)(x) = \int_{\mathfrak{X}} f(y)\, P(t, x, dy)$ and $G(x, E) = \int_0^\infty \exp(-t) P(t, x, E) dt$. *R. S. Phillips.*

Bochner, S. Partial ordering in the theory of martingales. Ann. of Math. (2) **62** (1955), 162–169.

The author generalizes the reviewer's concepts of semimartingale and martingale [Stochastic processes, Wiley, New York, 1953; MR **15**, 445] by allowing the parameter set of the stochastic process to be a directed set (rather than a linear set) and, sometimes, phrasing the definitions in terms of set functions rather than point functions. For example, a semimartingale is defined to be a triple $\{\Omega, \mathfrak{A}_\lambda, X_\lambda\}$, where λ varies on a directed set, $\mathfrak{A}_\lambda$ is an algebra of subsets of the space Ω, X_λ is a set function defined on $\mathfrak{A}_\lambda$, and both the class $\mathfrak{A}_\lambda$ and the function X_λ are monotone non-decreasing with λ. Further hypotheses are made to obtain specific results, but the stress is on unification rather than new theorems. It is noted that the set functions can be Hermitian matrix-valued rather than numerically valued. Some of the theorems are incorrect without stronger hypotheses, since they assume that a uniformly absolutely continuous family of set functions must be uniformly bounded. *J. L. Doob* (Urbana, Ill.).

Yaglom, A. M. Theory of extrapolation and filtering of random processes. Ukrain. Mat. Ž. **6** (1954), 43–57. (Russian)

Expository lecture, leading to the two papers reviewed below. *J. L. Doob* (Urbana, Ill.).

Yaglom, A. M. Extrapolation, interpolation and filtering of stationary random processes with rational spectral density. Trudy Moskov. Mat. Obšč. **4** (1955), 333–374. (Russian)

The author gives a very detailed discussion of his stated subject, distinguished by clear formulation and rigorous and explicit solutions. The results were extended to processes with stationary nth order increments in the paper reviewed below. The solutions were discussed in an earlier paper [see the preceding review]. *J. L. Doob.*

Yaglom, A. M. Correlation theory of processes with random stationary nth increments. Mat. Sb. N.S. **37(79)** (1955), 141–196. (Russian)

This paper contains a general survey of stochastic processes with (wide sense) stationary increments of order n (called "n-processes" below), extending the case $n=0$, for which the results are known. Examples are treated which have no analogue for the case $n=0$, however. In the first section, the spectral form of n-processes is obtained, in this generality due to Yaglom and Pinsker [Dokl. Akad. Nauk SSSR (N.S.) **90** (1953), 731–734; MR **15**, 238], and to Itô [Mem. Coll. Sci. Univ. Kyoto. Ser. A. **28** (1953), 209–223; MR **16**, 378].

The second section gives a complete solution of the problem of finding which n-processes can be written as moving averages, in the sense that their nth-order differences can be written in the form

$$\Delta_\tau^{(n)}\xi(t) = \int \Delta_\tau^{(n)} c(p) d_p \xi(t-p)$$

(over the whole or half p-axis). Here the ξ process has uncorrelated stationary (wide sense) increments. These results are applied to the extrapolation (prediction) problem, which is now that of finding the best (least squares) linear prediction for future nth-order increments in terms of past ones. The general results were stated in an earlier paper, by Pinsker and Yaglom [Dokl. Akad. Nauk SSSR (N.S.) **94** (1944), 385–388; MR **15**, 806].

The third section contains explicit solutions for extrapolation problems in various important special cases. The solution of the filtering problem when $n=0$ is extended to the general case, and examples are given.

The fourth section contains explicit solutions for extrapolation, interpolation and filtering problems for processes given in finite parameter intervals, assuming rational spectral densities. *J. L. Doob* (Urbana, Ill.).

Heine, V. Models for two-dimensional stationary stochastic processes. Biometrika **42** (1955), 170–178.

The results of Whittle [Biometrika **41** (1954), 434–449; MR **16**, 731] on the elliptic case are extended by a treatment of the general second-order linear stochastic partial

differential equation

$$\left(a\frac{\partial^2}{\partial x^2}+2h\frac{\partial^2}{\partial x\partial y}+b\frac{\partial^2}{\partial y^2}+2g\frac{\partial}{\partial x}+2f\frac{\partial}{\partial y}+c\right)\xi(x, y)=\varepsilon(x, y),$$

where ξ and ε are the variate and the random impulse effecting it. This leads to three types of model, corresponding to parabolic, elliptic and hyperbolic forms. Explicit expressions for the correlation and Green functions of the allowed standard models are derived and commented upon, with attention paid to degenerate cases. The methods follow van der Pol and Bremmer [Operational calculus based on the two-sided Laplace integral, Cambridge, 1950; MR **12**, 407]. In the parabolic case the x-axis is space-like, the y-axis time-like; when applied to diffusion phenomena this means that diffusion takes place in both directions of the x-axis, but in only one relative to the y-axis (which may e.g. represent distance downhill or downstream, or it may represent time during which something expands over an area). In the elliptic case both axes are space-like, and in the hyperbolic case both are time-like. *H. Wold* (Uppsala).

Du Mosch, A. D. On the average uncertainty of a continuous probability distribution. Appl. Sci. Res. B. **4** (1955), 469–473.

The author's aim is to modify Shannon's definition of the entropy, H, of an n-dimensional probability density [Bell System Tech. J. **27** (1948), 623–656; MR **10**, 133] by introducing a quantity $\bar{H}$ by means of an averaging process of H over a certain cell volume. Unlike H, $\bar{H}$ is necessarily non-negative, and is a measure of the degree in which the probability measure is spread out. The claim that $\bar{H}$ is independent of the coordinate system, however, would seem to require clarification. *E. Reich.*

Pignedoli, Antonio. Su alcuni problemi diffusivi della fisica matematica e su una questione connessa di meccanica statistica, interessante la teoria del moto browniano. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) **1** (1954), no. 2, 5–20.

The problems treated in this paper belong to the diffusion theory of neutrons in retarding media. The classical diffusion equation in ordinary space with a source term is first treated by the standard method of eigenfunction expansion. The diffusion equation in position and velocity space is considered next for the simplest case of a distribution function depending on one space coordinate x and one angular variable of the velocity. The solution is obtained in powers of x and t, t being the time, by the method of Laplace transforms. A discussion is finally given of the variation in the distribution of neutron energies, based on the assumptions that energy changes can be described as a random process governed by a Fokker-Planck equation and that there is an upper bound of energies never exceeded. No justification of this procedure is given. The paper does not take into account nor does it even quote the extremely abundant literature that has been published on the topics considered.

L. Van Hove (Utrecht).

Besse, L. Calcul des probabilités et applications à la balistique. Mém. Artill. Franç. **26** (1952), 381–476, 923–1003; **27** (1953), 97–129, 347–471.

This paper of 331 pages constitutes a thorough standard course in probability and statistics with applications to typical situations in exterior or interior ballistics. It treats sampling theory in a modern but practical manner and should replace older course material at the Ecole Nationale Supérieure de l'Armament. *A. A. Bennett.*

Have

★ **Saxer, Walter. Versicherungsmathematik. Teil I.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXIX. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. ix+249 pp. Geheftet DM 36.00; gebunden DM 39.60.

This is the first volume of a text on actuarial mathematics intended to replace A. Loewy's book with the same title [4th ed., Springer, Berlin, 1924] which has been out of print for many years. The book has a twofold purpose: to help in the academic training of actuaries and to serve as a reference work for actuarial practitioners. The present volume is quite elementary, while the second volume will deal with the more advanced parts of actuarial theory. The discontinuous method is used almost exclusively throughout the first volume; thus the use of calculus is avoided, the only exceptions being Chapter VIII and the appendix where certain special topics require some knowledge of elementary calculus. The first six chapters deal with the topics which any introductory text on actuarial mathematics must contain: Interest, Decrement tables [following A. Loewy's theory of intensities, S.-B. Heidelberger Akad. Wiss. **8A** (1917), no. 6], Annuities and Assurances on single and on joint lives, Pension and Group Insurance with strong emphases on the techniques needed in (compulsory) Social Insurance, Premium-reserves with due considerations of group methods used in actuarial practice. Chapter VII treats the variational problems of actuarial mathematics. The author follows here closely the general method of H. Schärf [Mitt. Verein. Schweiz. Versich.-Math. **41** (1941), 163–196; **43** (1943), 127–179; MR **6**, 94] which is applicable to the variation of all bases of calculation. Chapter VIII deals with the construction and application of universal tables; this is the only chapter in which the continuous method is used. This is an interesting and concise presentation of a topic which (as far as this reviewer knows) is not treated in any other recent actuarial textbook. Chapter IX discussed actuarial valuation and the distribution of dividends; Chapter X introduces the elements of renewal theory to the extent needed in Chapter XI which describes the systems of financing social insurance.

The book has a completely deterministic approach to actuarial mathematics. This means that it uses the scheme of a fictitious group of persons of equal age which decreases exactly in accordance with the mortality table. This approach is unrealistic but is in keeping with the actuarial tradition and is motivated by its simplicity. The author gives in the appendix a stochastic approach to actuarial mathematics which can be used to justify the deterministic procedure. Two stochastic models are discussed: the first is due to the reviewer [Trans. 12th Internat. Congress Actuar. Lucerne, 1940, vol. I, Füssli, Zurich, 1941, pp. 171–205; MR **3**, 177] and to S. Vajda [ibid., pp. 241–251, MR **3**, 177] while the second model was proposed by E. Sverdrup [Skand. Aktuarietidskr. **35** (1953), 115–131; MR **14**, 774]. The book is well written and gives a concise and clear introduction into the subject. This reviewer hopes that it will soon be followed by the second volume and that the complete work will offer a comprehensive and up-to-date survey of actuarial mathematics. *E. Lukacs* (Washington, D.C.).

See also: Davidson, McKinsey, and Suppes, p. 119; Huzurbazar, p. 120; Sundrum, p. 120; Spenser, p. 194; Mandelbrot, p. 170.

Mathematical Statistics

Masuyama, Motosaburo. Table of two-sided 5% and 1% control limits for individual observations of the r-th order. Sankhyā 15 (1955), 291–294.

Let $F(x)$ denote the distribution function of x, and $F(x_r)$ the distribution function of x_r, the r-th order statistic in a sample of n. The author finds the upper and lower limits U and L of $F(x_r)$ under the assumption of the continuity of $F(x)$ for equal tails $\alpha/2$. Tables of U and L are given for $n=3$ (1) 10, $r=1(1)$ n to 5 significant figures, $\alpha=.05$ and .01, and corresponding values when $F(x)$ is the normal distribution with mean zero and variance one. The results are useful in quality control. *L. A. Aroian.*

Leslie, P. H. A simple method of calculating the exact probability in 2×2 contingency tables with small marginal totals. Biometrika 42 (1955), 522–523.

Haldane, J. B. S. The rapid calculation of χ^2 as a test of homogeneity from a $2\times n$ table. Biometrika 42 (1955), 519–520.

James, G. S. Cumulants of a transformed variate. Biometrika 42 (1955), 529–531.

Farinha, João. Sur la moyenne arithmétique. Rev. Fac. Ci. Univ. Coimbra 23 (1954), 14–16.

It is shown that, for measurements $x_1, \cdots, x_n$ of a quantity whose true value is x, a "preferred" value $f(x_1, \cdots, x_n)$ necessarily is the arithmetic mean $\sum x_i/n$ provided f is differentiable, symmetric, and reflexive, and provided the difference $f-x$ is of the second order relative to the differences x_i-x. *E. F. Beckenbach.*

Majumer, Debidas. Frequency curves & curve fitting. J. Assoc. Appl. Phys. Calcutta Univ. 1 (1955), 84–95.

The author discusses Pearson's system of frequency curves. An example is presented of fitting a Type I curve to a sample of observations. The author does not differentiate between samples and populations and fails to indicate that Type I curves can be transformed to the simpler Incomplete Beta form, for which tables are available to compute fitted frequencies. No mention is made of methods of comparing observed and fitted frequencies. *R. L. Anderson* (Raleigh, N.C.).

Lehmann, E. L. Ordered families of distributions. Ann. Math. Statist. 26 (1955), 399–419.

Let $X=(X_1, \cdots, X_n)$ be a random vector with probability distribution P_θ depending on a real-valued parameter θ. In many problems the family of distributions is ordered by θ in the sense that large values of θ lead, on the whole, to large values of the X's. Several alternative definitions which make this notion precise are compared, and a number of applications are made. The three alternatives are: (A) For each pair, $\theta<\theta'$, there exists a random vector Z and functions $g=(g_1, \cdots, g_n)$ and $g'=(g_1', \cdots, g_n')$ such that $g_i(z)\leqq g_i'(z)$ $(i=1, \cdots, n)$ for all z, and the distributions of $g(Z)$ and $g'(Z)$ are P_θ and $P_{\theta'}$ respectively; (B) if $\theta<\theta'$, $P_\theta(S)\leqq P_{\theta'}(S)$ for every increasing set, S (a set in n-space is called "increasing" if $(x_1, \cdots, x_n)\in S$ and $x_i\leqq x_i'$ $(i=1, \cdots, n)$ implies $(x_1', \cdots, x_n')\in S$); (C) assuming that all of the distributions possess probability densities with respect to a common σ-finite measure, if $\theta<\theta'$ the ratio of the probability densities, $p_{\theta'}(x)/p_\theta(x)$ is an increasing function of $x=(x_1, \cdots, x_n)$ (a function φ defined over n-space is called "increasing" if for each pair x, x' with $x_i\leqq x_i'$ $(i=1, \cdots, n)$, $\varphi(x)\leqq\varphi(x')$). The combination (B) and (C) yields useful results (in general, (A)→(B), but if $X_1, \cdots, X_n$ are independent, (C)→(A)↔(B)) such as the monotonicity of the power of a sequential probability ratio test, and the identification of tests that give guaranteed power with a minimum number of observations. Applications are given to many interesting special cases. Some consequences of (B) and (C) are derived regarding the comparability of experiments in the sense of Blackwell [Proc. 2nd Berkeley Symposium Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 93–102; MR 13, 667]. An optimum property of the sign test is proved in an appendix.

Erratum: p. 414, line 16: for $p(x|\omega_1)\cdot p(x|\omega_2)$ read $p(x|\omega_1)/p(x|\omega_2)$. *P. Meier* (Baltimore, Md.).

Aitchison, John. On the distribution of a positive random variable having a discrete probability mass at the origin. J. Amer. Statist. Assoc. 50 (1955), 901–908.

Rider, Paul R. Truncated binomial and negative binomial distributions. J. Amer. Statist. Assoc. 50 (1955), 877–883.

Reissmann, Günter. Betrachtungen zu einem scheinbaren Widerspruch zwischen Fehlergesetz und Fehlerfortpflanzungsgesetz. Wiss. Z. Tech. Hochsch. Dresden 4 (1955), 597–600.

This is a very emphatic warning that the variance of a sum should not be confused with the variance of the average. The fact that the author considered it necessary to issue such a warning as well as the general flavor of the paper indicate that the methods of the German practical statisticians were not influenced by the mathematical theory of statistics developed outside Germany during the first half of this century. *E. Lukacs.*

Chu, John T. The 'inefficiency' of the sample median for many familiar symmetric distributions. Biometrika 42 (1955), 520–521.

Lord, Frederic M. Estimation of parameters from incomplete data. J. Amer. Statist. Assoc. 50 (1955), 870–876.

Banerjee, K. S. A note on successive sampling. Calcutta Statist. Assoc. Bull. 6 (1955), 35–39.

Let r_i denote the number of red balls obtained by an ith-stage sampling procedure, the population consisting of N_0p red balls and $N_0(1-p)$ white ones and the sampling procedure being at each stage random with or without replacement. (It is tacitly assumed that in cases of sampling with replacement an extra supply of balls is available, by means of which the successive samples can be constructed.) The author gives formulas for the variance of r_i for different cases. He shows that only two variance formulas are needed in the general case, namely one formula when sampling in the first stage is carried out with replacement and one formula when it is carried out without replacement. *D. M. Sandelius* (Göteborg).

Hammersley, J. M., and Nelder, J. A. Sampling from an isotropic Gaussian process. Proc. Cambridge Philos. Soc. **51** (1955), 652–662.

An isotropic gaussian process is defined. The authors are concerned with ways to insure that a physical process will have realizations which yield sample values for certain admissible isotropic gaussian processes. This is essentially a Monte Carlo problem where a prescription of a physical process is given. Then, assuming it will be actually possible to generate the physical process, attention is given to analytical questions such as determining what functions or scores of the physical process need to be considered. The reader can not expect to find a fully rigorous treatment of the underlying stochastic process. However, one does find attention given to characterizing solutions of certain integral equations which include the following generalization of Schlömilch's equation:

$$\frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\phi_n(t\sin\theta)\cos^n\theta\,d\theta=\Psi_n(t)\quad(0<t<\infty),$$

where n is a non-negative integer and Ψ_n is given.

M. Muller (Ithaca, N.Y.).

Walsh, John E. Approximate probability values for observed number of "successes" from statistically independent binomial events with unequal probabilities. Sankhyā **15** (1955), 281–290.

The author considers a sequence of alternatives with probabilities of success varying from trial to trial. Approximation formulae are given for the probability that the number of successes in n trials falls into a specified interval. The approximation is obtained by expanding the probability in terms of powers of the differences between the probabilities of success and their average. An outline of the derivations is also given. *E. Lukacs.*

Basu, D. An inconsistency of the method of maximum likelihood. Ann. Math. Statist. **26** (1955), 144–145.

Though it is known that the method of maximum likelihood may lead to an estimator which is inconsistent, the present author uses a pathological example to show that an inconsistent estimator is possible when all the observations are independently and identically distributed and involve only one unknown parameter. *M. Muller.*

Marakathavalli, N. Unbiased test for a specified value of the parameter in the non-central F distribution. Sankhyā **15** (1955), 321–330.

The author obtains an unbiased two-sided test of the hypothesis $\lambda=\lambda_0$ for the non-centrality parameter λ of the noncentral F-distribution (F'). Critical lower and upper values of F' are tabulated in special cases by means of an iterative method using Patnaik's F-approximation to F' [Biometrika **36** (1949), 202–232; MR **11**, 608]. Several applications are considered. *H. A. David* (Melbourne).

Des Raj. On sampling with varying probabilities in multistage designs. Ganita **5** (1954), 45–51.

Kempthorne, Oscar. The randomization theory of experimental inference. J. Amer. Statist. Assoc. **50** (1955), 946–967.

The use of randomization in experimental designs permits statistical inferences which depend on relatively few assumptions. A general mathematical treatment of randomization theory is presented and exemplified by some common designs. Randomization theory is compared with the classical (Gauss-Markoff theorem) approach from the point of view of justifying the latter by its approximate equivalence, in certain circumstances, to the former. The equivalence is shown to be fairly close in some special cases of importance, provided additivity(as defined in the paper) holds. *P. Meier* (Baltimore, Md.).

Rudra, A. A method of discrimination in time series analysis. I. Sankhyā **15** (1955), 9–34.

A review and development of the time-series model

$$x_t=x+\sum_1^k\theta_i^{(t)}(p_i)+\varepsilon_t\quad(t=1,\cdots,n),$$

where x is constant, ε_t a residual, and $\theta t^{(i)}$ a component with period p_i and with unspecial values $\theta^{(i)}(1),\cdots,\theta^{(i)}(p_i)$. The significance of a periodicity p_i is tested by a variance ratio F. For one component and with a random series for null hypothesis, F is distributed as a ratio of two independent noncentral chi-squares. Detailed illustrations, covering 16 artificial time-series, 6 of which are constructed by the author, and 22 natural time-series treated in earlier literature. Heuristic and partly misleading comments on the general cases of (a) several components, and (b) moving summations of random series for null hypothesis; thus section 5.2 contradicts Whittle's result that the expectation of the largest spurious peak in a Schuster periodogram is of order $2n^{-1}\log\frac{1}{2}n$, even in case (b) [see his Appendix 2 in H. Wold, A study in the analysis of stationary time series, 2nd ed., Almqvist-Wiksell, Stockholm, 1954; MR **15**, 811]. *H. Wold.*

★ **Mandelbrot, Benoit. On recurrent noise limiting coding.** Proceedings of the symposium on information networks. New York, April, 1954, pp. 205–221. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

Let $\mathcal{F}$ be a class of distinct finite sequences of letters constructed from an alphabet of M letters. Let $S(C)=$the number of sequences in $\mathcal{F}$ of length C. Assume that $\mathcal{F}$ satisfies the condition (R): If $\sigma\in\mathcal{F}$, then a sequence formed by truncating σ is not in $\mathcal{F}$. It then follows, as pointed out by Szilard, that

$$(1)\qquad \sum_{C=1}^{\infty}S(C)M^{-C}\leq 1.$$

In the author's terminology an element of $\mathcal{F}$ is the encoded form of a "word," and $S(C)$ is the "structure function" of the code. The condition (R) corresponds to the code's being "recurrent." He shows that (1) continues to hold for certain non-recurrent types of coding, and also discusses other related problems. The investigation is partially connected with the author's theory of language [cf. Communication theory (W. Jackson, ed.), Academic Press, New York, 1953, pp. 486–502]. *E. Reich.*

★ **Dumas, Maurice. Les épreuves sur échantillon.** Centre National de la Recherche Scientifique, Paris, 1955. 170 pp. 1000 francs.

A manual of statistical methods for engineers and physicists, contains concisely formulated recipes for solution of a number of problems of estimation, of tests of statistical hypotheses and of control of quality. Also, there is a brief theoretical section intended to introduce the reader to general statistical concepts. *J. Neyman.*

Theory of Games, Mathematical Economics

Eng
L16
★ **Bellman, Richard. Dynamic programming of continuous processes.** The Rand Corporation, Santa Monica, Calif., 1954. xviii+141 pp.

Chapter I of this report is preliminary. Chapter II gives a complete discussion of the problem of finding the ϕ_1 which maximizes $f(T)$, where $0 \leq \phi_1(t) \leq 1$, and f is determined by the differential equations

$$(1)\quad \begin{aligned} dx/dt &= -\phi_1(t)r_1x(t), & x(0)&=x_0,\\ dy/dt &= -\phi_2(t)r_2y(t), & y(0)&=y_0,\\ dp/dt &= -p(t)[\phi_1(t)q_1+\phi_2(t)q_2], & p(0)&=1,\\ df/dt &= p(t)[\phi_1(t)rx(t)+\phi_2(t)r_2y(t)], & f(0)&=0, \end{aligned}$$

where $0 \leq t \leq T \leq \infty$ and $\phi_2(t)=1-\phi_1(t)$. The author is able to find $\phi_1(t)$ explicitly. He continues with a discussion of a logically substantially more complicated problem in which three variable functions ϕ_1, ϕ_2, and ϕ_3 appear, and of a nonlinear analogue of the system (1). These problems arose from decision problems in which decisions were made at discrete time intervals. The author devotes Chapter III to a proof that the "return" function $f(T)$ is close to the return functions of the approximated problems when the time intervals are short.

Chapters IV and V discuss "bottleneck problems," which lead to systems of linear ordinary differential equations in which the variables or linear combinations of the variables are subjected to inequality bounds. These have been sketched previously by the author [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 947–951; MR **15**, 888]. Here several complicated examples are worked out in detail.

Chapters VI and VII (the latter by H. Osborn) contain a discussion of the relation of dynamic programming to the classical calculus of variations. A portion of the material has been published before [ibid. **40** (1954), 231–235; MR **15**, 804]. Chapter VIII discusses scheduling problems, for which see the review [MR **16**, 843] of a paper by Bellman and Gross [J. Soc. Indust. Appl. Math. **2** (1955), 175–183].

J. M. Danskin (Princeton, N.J.).

Dantzig, George B. Optimal solution of a dynamic Leontief model with substitution. Econometrica **23** (1955), 295–302.

If the unknown quantities of activities are to be determined so as to minimize a specified linear objective function in a generalized Leontief system with substitution under some conditions, the following propositions are established: For $\sum a_{ij}x_j=b_i$ where $b_i>0$ and (a_{ij}) can be partitioned into "block trianglar" form (α_{ij}) $(\alpha_{ij}=0$ if $i<j$; $\alpha_{ij}\neq 0$ if $i\geq j)$, where $\alpha_{t\tau}$ is a submatrix of (a_{ij}) with m_t rows and n_τ columns such that for $t\neq\tau$ all elements of $\alpha_{t\tau}$ are negative or zero, and such that diagonal submatrices α_{tt} contain exactly one positive element in each column, (1) x_j are independent of b_i; (2) the selection of first, say, k activities can be made without consideration of the following activities in x_j and so forth. Some economic interpretations of great interest are given to these propositions.

S. Ichimura (Osaka).

Inada, Ken-ichi. Alternative incompatible conditions for a social welfare function. Econometrica **23** (1955), 396–399.

K. Arrow [Social choice and individual values, Wiley, New York, 1951; MR **12**, 624] has defined a social-welfare function as a rule which attaches to each collection of individual orderings of a set of alternatives a social ordering of this set. He then gives a set of five conditions which such a function is required to satisfy and shows that the conditions are incompatible. In the present paper the author replaces these conditions by four weaker conditions (implied by but not implying Arrow's conditions) and shows that in this case too, the social welfare function does not exist.

D. Gale (Providence, R.I.).

Morin, François. Note on an inventory problem. Econometrica **23** (1955), 447–450.

Let the nonnegative, nondecreasing, piecewise continuously differentiable function $S(t)$ be the given schedule of cumulative sales over the interval $(0, t)$. Let $f(x)$ be the cost per unit time of producing at the rate x units of output per unit time $(f', f''>0)$, h_0 the given initial inventory, α the cost of storage per unit of product per unit time, and $X(t)$ the unknown (but sharing the properties of S) production schedule, defined as h_0+cumulative output up to time t. Thus $X-S$ is inventory at time t. The problem is to choose $X(t)$ to minimize total cost$=\int_0^T[\alpha(X-S)+f(\dot X)]dt$ subject to the boundary conditions: $X(0)=h_0$, $X(T)=S(T)$ (no terminal inventory), and $X(t)\geq S(t)$ $(0\leq t\leq T)$. The Euler equation $\dot X f''(\dot X)=\alpha$ contains neither X nor t explicitly, hence the pieces of the extremals interior to the constraint region are of the form $X-\bar X=\Phi(t-\bar t)$. The usual corner condition and the one-sided requirement that $\dot S f''(\dot S)\geq\alpha$ whenever a minimizing arc coincides with the boundary together yield further qualitative properties of the solution. This problem is closely related to one treated by Modigliani and Hohn [Econometrica **23** (1955) 46–66; MR **16**, 733], except that the latter deal in discrete unit intervals of time with sales and output rates constant within periods.

R. Solow (Cambridge, Mass.).

★ **Menger, K. The logic of the laws of return. A study in meta-economics.** Economic activity analysis, pp. 419–482. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London. $6.75. Have

What is called the law of diminishing return in economics is precisely formulated, and its associated propositions and their mutual relations are carefully clarified. The paper is a revised version of Z. Nationalökonomie **7** (1936), 25–56, 388–397.

S. Ichimura (Osaka).

See also: Davis, p. 197.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

★ **Pontryagin. L. S. Nepreryvnye gruppy. [Continuous groups.]** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 515 pp. 19.40 rubles.

This new edition of Pontrjagin's book [1st ed., GTTI, Moscow, 1938; English translation, Princeton, 1939; MR **1**, 44] on topological groups differs from the first by the many changes and augmentations of which the principal ones are: Addition of a new chapter on the classification of compact Lie algebras; insertion of a separate chapter on topological rings and fields; replacement of the locally compact separable groups which were considered throughout in the first edition by locally bicompact groups (and

which we will call locally compact, sticking to the current terminology); addition of many new examples which illustrate or elaborate particular points of the general theory.

In chapter I, which discusses the basic concepts of group theory, a discussion of direct products (weak and strong) along with a discussion of the rank concept for non-finitely generated abelian groups is included in view of later needs. Apart from examples to these new topics a number of examples have been added to show how groups arise very naturally as transformation groups in geometry. The chapter concludes with a discussion of the ring and field concept and the connection between the latter and projective geometry (Desargues' and Pascal's theorems).

Chapter II is a modern standard treatment of the fundamental topological notions like complete regularity, compactness, theorem of Tychonoff, connectedness, dimension. Some attention is paid also to the notion of weight=least cardinal of the open bases of the space.

Chapter III which considers topological groups in general has been expanded by a paragraph on transformation groups. Among the theorems proved here we mention: a locally compact group acting effectively and transitively on a locally compact Hausdorff space has but one locally compact topology compatible with the way in which the group operates on the space.

Chapter IV is entirely new and is mainly devoted to the proof of a general structure theorem for locally compact nondiscrete fields due to Kowalsky and which says that any such field is a finite extension either of the reals, or of a p-adic field, or of the field of formal power series over the prime field of characteristic $p>0$.

Chapter V develops integration on compact groups by von Neumann's method of means and proves the Peter-Weyl theorem. It is essentially a modernized version of chapter IV in the old edition. In the next chapter which gives an exposition of the theory of characters of locally compact abelian groups, questions of connectedness are considered in more detail than before (e.g., it is proved that dimension of compact abelian group=rank of character group; criteria for connectedness and local connectedness in terms of character group properties are given). Here also the weight concept finds some application.

The four following chapters are, except for some changes, the chapters VI–IX in the first edition. In chapter VIII some material on compact transformation groups is included (it is proved e.g. that any compact connected group acting effectively and transitively on a finite-dimensional locally connected space is a Lie group), and the chapter on covering spaces has been enriched by a number of examples.

Chapter eleven, the last one, gives an exposition of the structure theory of compact Lie algebras on the basis of the classical Killing-Cartan-Weyl theory. It concludes with an enumeration of the simple root systems after the method of Dynkin and gives their Coxeter diagrams explicitly.

The reviewer would have liked more references to work that has been done or is being done on subjects closely related to those treated in this book and which could not be reported on here. Despite this, however, the book seems to the reviewer to be still the best general introduction at present to the theory of topological groups. *W. T. van Est.*

Isiwata, Takesi. On the connectedness of locally compact groups. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 8–13.

The main interest of this paper lies in the fact that it is one of the few concerned with the existence of locally compact topologies on a commutative group comparable with a given topology. The author also proves theorems (some known) relating properties of locally compact groups with existence, or not, of characters which are open or closed. *M. E. Shanks* (Lafayette, Ind.).

Enomoto, Shizu. Sur la structure des fonctions d'ensemble dans les groupes topologiques localement compacts. I. Proc. Japan Acad. 31 (1955), 284–287.

The author proves the following well known result: If G is a locally compact, σ-compact, topological group and θ_n is a sequence of compact neighborhoods such that $\theta_n\theta_n^{-1}\subset\theta_{n-1}$, then $G_b=\bigcap_n \theta_n$ is a compact invariant subgroup such that G/G_b is a separable, metric, locally compact group. *P. S. Mostert* (New Orleans, La.).

Hewitt, Edwin, and Rubin, Herman. The maximum value of a Fourier-Stieltjes transform. Math. Scand. 3 (1955), 97–102.

Let G be a locally compact abelian group with dual G^*, and let (y, x) denote the value at $x\in G$ of the character $y\in G^*$. If φ is a bounded Radon measure on G, and if $\Phi(y)=\int_G (y, x)d\varphi(x)$ is its Fourier transform, it is trivial that

$$|\Phi(y)|\leq\int_G d|\varphi|(x),$$

$|\varphi|$ being the total variation of φ. For a given φ, denote by $A(\varphi)$ and $M(\varphi)$ respectively the set of $y\in G^*$, where

$$|\Phi(y)|=\int_G d|\varphi|(x) \text{ and } \Phi(y)=\int_G d|\varphi|(x).$$

The principal result of the paper under review asserts the equivalence of the following three conditions on a set $E\subset G^*$: (1) E is an $A(\varphi)$ for some φ; (2) E is an $M(\varphi)$ for some φ; (3) E is either void, or is a translate of a closed subgroup of G^* and contains a non-void G_δ. In addition, if E satisfies one of these conditions, φ can be chosen absolutely continuous relative to Haar measure on G if and only if E is compact. The proofs are simple and ingenious. *R. E. Edwards* (London).

Wallace, A. D. The position of C-sets in semigroups. Proc. Amer. Math. Soc. 6 (1955), 639–642.

A C-set of a continuum (i.e., a compact connected Hausdorff space) is a proper subset C of X such that for each subcontinuum A of X that meets C, either $A\subset C$ or $C\subset A$. Throughout this review S will denote a compact connected Hausdorff topological semigroup. It is known [Numakura, Math. J. Okayama Univ. 1 (1952), 99–108; MR 14, 18] that the set E of idempotents of S is non-empty, and that S has a minimal (two-sided) ideal K, which is closed. Moreover, if for $e\in E$, $H(e)$ denotes the maximal subgroup of S containing E, then $K=\bigcup\{H(e): e\in E\cap K\}$.

In this note, the author generalizes some of his previous work [ibid. 3 (1953), 1–3; MR 15, 933]. The main results are: If S has a left unit, then for any closed left ideal of S and any C-set C of S, if C meets $K\cup L$, then $C\subset K\cup L$. If S has a unit, and if K is a C-set, then K is a maximal subgroup of S. If S has a unit, $e\in E$ is such that $H(e)\neq\{e\}$, and if C is a C-set of S meeting $H(e)$, then $C\subset H(e)$. If S has a unit u, and if $H(u)$ has a non-void interior, then $H(u)=S$. For any S, if $e\in E\cap K$, and $H(e)$ has a non-void interior, then $H(e)=K$. [An obvious misprint occurs in the statement of Theorem 3: "W" should read "$\neq$".] *M. Henriksen* (Lafayette, Ind.).

Švarc [Schwarz], Štefan. On expanding elements in the theory of semigroups. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 697–698. (Russian)

Let S be a semigroup. An element $t \in S$ is called a left-expanding element if there is a proper subset $M \subset S$ such that $Mt=S$. Similarly, one defines right-expanding elements. The author proves that if S is a compact Hausdorff semigroup then S can contain no left or right expanding elements. (A. D. Wallace [Math. J. Okayama Univ. 3 (1953), 23–28; MR 15, 933] has shown that if S is a compact Hausdorff semigroup, and A is a closed subset, and $A \subset At$ for some t, then $A=At$, and the function $f(x)=xt$ is a homeomorphism on A. A slight extension of his argument yields the author's result.) *A. L. Shields.*

Iséki, Kiyoshi. Sur un théorème de M. G. Thierrin concernant demi-groupe limitatif. Proc. Japan Acad. 31 (1955), 54–55.

A semigroup (Fr. "demi-groupe") is called limitative by G. Thierrin [C. R. Acad. Sci. Paris 238 (1954), 1765–1767; MR 15, 849] if $ax=bx=a$ implies $a=b$ and left-right dually. The author shows that if a compact Hausdorff topological semigroup D is limitative then D is a group. *A. H. Clifford* (New Orleans, La.).

Faucett, W. M. Compact semigroups irreducibly connected between two idempotents. Proc. Amer. Math. Soc. 6 (1955), 741–747.

Throughout this review, S will denote a compact (Hausdorff) connected topological semigroup, and I will denote the closed interval [0, 1] of the real line under the usual multiplication. If S is irreducibly connected between two of its elements (i.e., if there is no proper connected subset of S containing both of them), then S can be linearly ordered, and its topology is equivalent to the interval topology.

The main results of this note are: If S is irreducibly connected between two idempotents, and contains no other idempotents, then S is abelian and homeomorphic to I. If, in addition, S contains no (proper) nilpotent elements, then a homeomorphism of S upon I can be chosen to be an isomorphism as well. If S has a unit element, and is irreducibly connected between two idempotents, then S is abelian if and only if it has a zero-element. More generally, if S is irreducibly connected between two idempotents e and f, then S is abelian if and only if S has a zero-element and $ef=fe$. Finally, examples are given, which show that the hypothesis of these theorems cannot be weakened in certain directions.

M. Henriksen (Lafayette, Ind.).

Faucett, W. M. Topological semigroups and continua with cut points. Proc. Amer. Math. Soc. 6 (1955), 748–756.

As in the preceding review, S denotes a compact (Hausdorff) topological semigroup. [For background on compact topological semigroups, see Wallace, Bull. Amer. Math. Soc. 61 (1955), 95–112; MR 16, 796; for other definitions, see Whyburn, Analytic topology, Amer. Math. Soc. Colloq. Publ., v. 28, New York, 1942; MR 4, 86.] Only the main results of the present paper are given below.

If p is a cutpoint of the minimal closed (two-sided) ideal K of S, then either $K=Sp$, K is a minimal left ideal and every element of K is a left zero for S, or $K=pS$, K is a minimal right ideal, and every element of K is a right zero for S. If $S^2=S$, and if J is a maximal proper ideal of S, then for any closed subset M of $S\setminus J$, the set $S\setminus M$ is connected ($\setminus$ denotes set-theoretic difference.)

If a closed subset N of S such that the closure of its complement is neither all of S nor meets N in more than one point, then N is called a nodal set. A D-chain A is a non-empty subset that is the intersection of all the nodal sets containing it. It is shown that if C is both a D-chain and a subsemigroup of S, that is contained in $S\setminus K$, then there is an x in C such that $xC=x=Cx$. In the final section of the paper, it is shown that if S has a unit, and is a metric tree (dendrite), and if the set N of endpoints of S is contained in the center of S, then S is abelian and has a zero. If S is a metric tree, and N is a group, then S has a zero. If, in addition, N is abelian, then S is abelian. [Remark: In Theorem 1.2, $F(L)$ seems to denote the boundary of L.] *M. Henriksen* (Lafayette, Ind.).

See also: Ehrenpreis and Mautner, p. 126.

Lie Groups, Lie Algebras

Gel'fand, I. M., and Graev, M. I. Analogue of the Plancherel formula for the classical groups. Trudy Moskov. Mat. Obšč. 4 (1955), 375–404. (Russian)

In the proof of the Plancherel formula for the complex unimodular group G given by Gelfand and Neumark [Trudy Mat. Inst. Steklov. 36 (1950); MR 13, 722] the principal step consisted in establishing the formula $x(e)=(L(I_\delta))_{\delta=e}$ where x is any member of a certain class of complex-valued functions on G, e is the identity of G, $\delta \rightarrow I_\delta$ is a complex-valued function on the set of diagonal matrices δ in G defined for each δ by integrating x over the class of elements conjugate to δ, and L is a certain differential operator which is independent of x. Recently the present authors [Dokl. Akad. Nauk SSSR (N.S.) 92 (1953), 221–224; MR 15, 601] sketched a method for obtaining such a formula which was not only simpler than that employed by Gelfand and Neumark but also could be applied to the other classical groups as well. The paper under review is devoted to giving the details for the complex unimodular group, the orthogonal group and the symplectic group. The method depends upon making use of suitable generalizations of recent results of M. Riesz [Acta Math. 81 (1949), 1–223; MR 10, 713] and the first section of the paper is concerned with the development of these generalizations. Harish-Chandra has recently given a derivation, along quite different lines, of the Plancherel formula for an arbitrary complex semi-simple Lie group [Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 813–818; Trans. Amer. Math. Soc. 76 (1954), 485–528; MR 13, 533; 16, 111.] *G. W. Mackey.*

Harish-Chandra. On the characters of a semisimple Lie group. Bull. Amer. Math. Soc. 61 (1955), 389–396.

Let G be a connected semisimple Lie group and π a representation of G by bounded operators on a Hilbert space H. Assume π is quasi-simple in the sense of the author (the center of G and of the associative enveloping algebra of the Lie algebra of G are mapped onto scalars). Let $f(x)$ be an indefinitely differentiable function of compact support on G; then the bounded operator $\int_G f(x)\pi(x)dx$ has a trace $T_\pi(f)$ which is a distribution in the sense of L. Schwartz, as has been proved by the author previously [Trans. Amer. Math. Soc. 76 (1954), 234–253; MR 16, 11]. The main purpose of this paper

is to study this distribution in more detail. Let G' be the set of regular elements of G. Theorem 1 asserts that there exists an analytic function F_π on G' such that $T_\pi = F_\pi$ on G'. Next a formula is derived for F_π which generalizes H. Weyl's formula for the characters of the compact semisimple Lie groups. This formula also contains the expression derived for F_π by Gelfand and Naimark for the complex unimodular group. Finally it is proved that if π_0 is a quasi-simple irreducible representation of G for which F_{π_0} is not identically zero, then there exist (up to infinitesimal equivalence) only a finite number of quasi-simple irreducible representations π such that $F_\pi = F_{\pi_0}$. *F. I. Mautner* (Princeton, N.J.).

Patterson, E. M. Note on three-dimensional Lie groups. Proc. Glasgow Math. Assoc. 2 (1955), 112–115.

The author displays representatives of all the isomorphism types of real three-dimensional Lie algebras and of the local isomorphism types of the corresponding Lie groups. The equivalence relation (which is essentially congruence) among multiplication table matrices defined by isomorphism of the algebras is determined and a set of representatives chosen. Apart from the two simple algebras, the nilpotent algebra and the null algebra, four classes of solvable algebras appear. The Lie groups are obtained from the groups of automorphisms leaving invariant the bilinear forms determined by the matrices in the representative set. *W. G. Lister.*

Dieudonné, Jean. Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique $p>0$. III. Math. Z. 63 (1955), 53–75.

[For parts I and II see Comment. Math. Helv. 28 (1954), 87–118; Amer. J. Math. 11 (1955), 218–244; MR 16, 12, 789.] Let p be a prime, and let x_i be the isobaric component of weight p^i of the formal power series $\exp(u_0+u_1+\cdots)$ in the sequence of variables u_k carrying the weight p^k. Then $\exp(u_0+u_1+\cdots)=E(x_0, x_1, \cdots)$, where E is a certain power series with rational coefficients, which turn out to be p-adic integers. Replacing these coefficients by their residue classes in the prime field F_p of characteristic p, one obtains the "hyperexponential series" $\mathrm{Hex}(x_0, x_1, \cdots)$, with coefficients in F_p. The multiplication of exponentials is then reflected in an abelian formal group law $\varphi_i(x, y)=z_i$, where z_i is the isobaric component of weight p^i (x_k and y_k carrying the weight p^k) of

$$\mathrm{Hex}(x_0, x_1, \cdots)\,\mathrm{Hex}(y_0, y_1, \cdots)=\mathrm{Hex}(z_0, z_1, \cdots).$$

This group law is "recursive", in the sense that z_i depends only on the x_k and y_k with $k\leq i$. The formal group defined by φ is called the hyperexponential group and is denoted H.

The hyperalgebra of H is determined explicitly in terms of a standard basis and turns out to be a polynomial algebra. An abelian recursive formal group G is said to be of hyperexponential type of its Lie algebra (a small part of the hyperalgebra) is the same as that of H and if its hyperalgebra furthermore satisfies a certain formal regularity condition. It is shown that if G is of hyperexponential type there exists a recursive isomorphism of G onto H. In particular, the formal group corresponding to the addition of Witt vectors, and also that corresponding to the multiplication of Witt vectors with first components equal to 1, is of hyperexponential type.

The author then defines a certain "universal group", U, whose hyperalgebra is the same as that of H, and shows how to obtain a large variety of abelian formal Lie groups as 'factor groups' of U. *G. P. Hochschild.*

Dieudonné, Jean. Lie groups and Lie hyperalgebras over a field of characteristic $p>0$. IV. Amer. J. Math. 77 (1955), 429–452.

Let K be a field of characteristic $p\neq 0$, and let W_K denote the ring of the Witt vectors $(a_0, a_1, \cdots)$, with the a_i in K. The addition in W_K is defined by a recursive formal group law $\phi=(\phi_0, \phi_1, \cdots)$ consisting of certain formal integral power series ϕ_i in the two sequences of variables $x=(x_0, x_1, \cdots)$, $y=(y_0, y_1, \cdots)$, with coefficients in the prime field of characteristic p, such that $\phi_i(x, y)=x_i+y_i+\psi_i(x_0, y_0, \cdots, x_{i-1}, y_{i-1})$, the power series ψ_i being isobaric of total weight p^i when x_k and y_k are assigned the weight p^k. A recursive K-endomorphism of the corresponding formal group W (regarded as a formal Lie group over K) is a sequence $u=(u_0, u_1, \cdots)$ of formal integral power series u_i in x, with coefficients in K, such that $u_i(0)=0$, $u_i(x)$ involves only $x_0, x_1, \cdots, x_i$, and $\phi(u(x), u(y))=u(\phi(x, y))$. These recursive endomorphisms constitute a ring E, the product being the composition of endomorphisms, and the sum being defined by $(u+v)(x)=\phi(u(x), v(x))$.

Using also the formal multiplication law in W, one may identify every Witt vector A in W_K with the K-endomorphism $x\rightarrow Ax$ of W, so that W_K becomes identified with a subring of E. There are two special endomorphisms of W that are of intrinsic significance: the "Frobenius endomorphism" P, defined by $P_i(x)=x_i^p$, and the "shift endomorphism" T, defined by $T_0(x)=0$, and, for $i>0$, $T_i(x)=x_{i-1}$. The author shows first that, if K is perfect, every u in E can be written uniquely in the form $u=\sum_{k=0}^\infty (A_kP^k+T^kB_k)$, with the A_k and B_k in W_K.

If $B=(b_0, b_1, \cdots)$ is any element of W_K one has $BT=TB^\sigma$, where $B^\sigma=(b_0^p, b_1^p, \cdots)$. Hence the endomorphisms of the form $\sum_{k=0}^\infty T^kB_k$ constitute a subring F of E. Using results of part III (reviewed above) and in particular his hyperexponential group (which is a certain recursive formal group, recursively isomorphic with W) the author associates with every n by n ($n=1, 2, \cdots$) matrix U with elements in F a certain hyper Lie algebra, and proves that this is actually the hyper Lie algebra of a certain abelian n-dimensional formal Lie group G_U over K. Moreover, he shows that every abelian formal Lie group is isomorphic with such a G_U.

Furthermore, assuming that K is perfect, the author connects the homomorphisms of an n-dimensional G_U into an m-dimensional G_V with n-by-m matrices with elements in F. The results are finally stated in module-theoretical form: let F^n denote the free F-module of rank n, let I_n stand for the n-by-n identity matrix, and regard TPI_n-TU as an F-endomorphism of F^n, sending F^n onto a submodule U' of F^n. Then the homomorphisms of G_U into G_V correspond exactly to the F-homomorphisms of F^n/U' into F^m/V'. In particular, (now $n=m$) G_U and G_V are isomorphic if and only if the F-modules F^n/U' and F^n/V' are isomorphic, or if and only if the matrices TPI_n-TU and TPI_n-TV are equivalent.

One of the applications made by the author is to show that there is a formal abelian Lie group G of dimension 2 which is "analytically simple", in the sense that every homomorphism of a 1-dimensional group into G is trivial. *G. P. Hochschild* (Berkeley, Calif.).

Dieudonné, Jean. Witt groups and hyperexponential groups. Mathematika 2 (1955), 21–31.

The formal group W obtained from the addition of Witt vectors is studied from the point of view of hyperalgebras of formal groups, and an explicit isomorphism of W onto the author's hyperexponential group (see the paper reviewed second above) is given.

G. P. Hochschild (Berkeley, Calif.).

Cohn, P. M. A non-nilpotent Lie ring satisfying the Engel condition and a non-nilpotent Engel group. Proc. Cambridge Philos. Soc. 51 (1955), 401–405.

This paper is devoted to the construction of a Lie algebra L over K^* of characteristic p with the property that $(\mathrm{ad}\, x)^{p+1}=0$ for each x in L, and L is solvable but not nilpotent. L can be described roughly as follows: let K be a countably transcendent extension of the prime field, M a certain associative bi-module for K, A the split-null extension of K determined by M, and K^* the center of A. L is the Lie algebra of A.

W. G. Lister (Providence, R.I.).

Jacobson, N. Commutative restricted Lie algebras. Proc. Amer. Math. Soc. 6 (1955), 476–481.

If $\mathfrak{M}$ is a semi-simple restricted Lie algebra over a field Φ of characteristic p, the decomposition of $\mathfrak{M}$ relative to a Cartan subalgebra $\mathfrak{L}$ presents the problem of studying the commutative restricted subalgebras $\mathfrak{L}$ of $\mathfrak{M}$. If Φ is algebraically closed, this amounts to the study of the semi-linear transformation $T: a\to a^p$ in $\mathfrak{L}$. The elementary divisor decomposition of $\mathfrak{L}$ with respect to T is used to show: i) if $\mathfrak{L}$ is semi-simple then $\mathfrak{L}$ is cyclic relative to T; ii) if $\mathfrak{L}$ is simple, $\mathfrak{L}=[h]$ is one dimensional, and if $T\neq 0$, h can be chosen so that $h^p=h$; iii) if $\mathfrak{L}$ is semi-simple, it is the direct sum of simple subalgebras in each of which $T\neq 0$. The ring of endomorphisms of $\mathfrak{L}$ is a complete matrix algebra over the prime field.

W. G. Lister (Providence, R.I.).

Jenner, W. E. Arithmetics of Lie algebras. Nieuw Arch. Wisk. (3) 3 (1955), 72–78.

Let $\mathfrak{o}$ be a Dedekind ring of characteristic 0 with quotient field k, and let $\mathfrak{L}$ be a semisimple Lie algebra of finite dimension over k. A subring $\mathfrak{O}$ of $\mathfrak{L}$ is called integral if it is a finitely generated $\mathfrak{o}$-module containing a k-basis for $\mathfrak{L}$. The discriminant of $\mathfrak{O}$ is defined in analogy with the discriminant of E. Cartan for associative algebras. If $\mathfrak{D}$ and $\mathfrak{D}^*$ are the discriminants of integral subrings $\mathfrak{O}$ and $\mathfrak{O}^*$ resp. with $\mathfrak{O}\subseteq\mathfrak{O}^*$ then $\mathfrak{O}\subset\mathfrak{O}^*$ if and only if $\mathfrak{D}\subset\mathfrak{D}^*$. Hence any integral subring of $\mathfrak{L}$ can be embedded into a maximal integral subring. Let $\mathfrak{p}$ be a prime ideal of k. If $\mathfrak{p}$ does not divide $\mathfrak{D}$, then $\mathfrak{O}-\mathfrak{p}\mathfrak{O}$, as an algebra over $\mathfrak{o}/\mathfrak{p}$, is a direct sum of simple non-abelian Lie algebras. If $\mathfrak{o}_p$ denotes the ring of local integers at $\mathfrak{p}$ then $\mathfrak{o}_p\mathfrak{O}$ is a multiplicative unit operator on its ideals for almost all $\mathfrak{p}$. If $\mathfrak{O}$ is a Lie ring which is a multiplicative unit operator on its ideals, a direct-intersection ideal theory can be developed along the lines of a previous paper [Compositio Math. 11 (1953), 187–203; MR 16, 7]. Then every ideal of $\mathfrak{O}$ has a unique representation as a direct intersection of block ideals provided that the ascending chain condition holds for ideals of $\mathfrak{O}$.

A. Jaeger.

See also: Kostrikin, p. 126.

Topological Vector Spaces

Schaefer, Helmut. Über die Methode der a priori-Schranken. Math. Ann. 129 (1955), 415–416.

It is one of the consequences of the Leray-Schauder theory that for a certain class of functional equations depending on a parameter the existence of a solution is assured once one knows that the possible solutions are bounded. In the present paper this fact is proved without use of the Leray-Schauder theory using Tychonoff's fixed-point theorem. The proof is carried through for equations of the form $X=\lambda\varphi(X)$ where X denotes a point of a linear locally convex complete Hausdorff space E and φ is a completely continuous map of E into itself while λ is a real-valued parameter varying in the closed unit interval.

E. H. Rothe (Ann Arbor, Mich.).

Iséki, Kiyoshi. Vector-space valued functions on semigroups. II. Proc. Japan Acad. 31 (1955), 152–155.

Two distinct types of results are obtained in this paper which is a continuation of an earlier one [same Proc. 31 (1955), 16–19; MR 16, 1030]. The gist of those of the first type is that the mean value is a linear operator on any linear space of ergodic vector-valued functions on a semi-group. The conclusion in the second type is that an almost periodic function on a semi-group is ergodic.

B. R. Gelbaum (Minneapolis, Minn.).

Marinescu, G. Équations transposées dans la théorie des distributions. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 65–74. (Romanian. Russian and French summaries)

The author modifies slightly the definition of fundamental space given by Gel'fand and Šilov [Uspehi Mat. Nauk (N.S.) 8 (1953), no. 6(58), 3–54; MR 15, 867]. He repeats the proof that derivatives can be defined in the space of distributions for his case. For L. Schwartz's spaces $\mathcal{D}$, $\mathcal{D}^*$, and $\mathcal{E}$ [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR 12, 31, 833], the following theorem is proved. Let $\alpha\in\mathcal{E}$, $\varphi\in\mathcal{D}$, and let $\psi=\alpha\varphi$. Suppose that $\psi\to 0$ in $\mathcal{D}$ implies that $\varphi\to 0$ in $\mathcal{D}$. Then for every distribution $S\in\mathcal{D}^*$, the equation $\alpha T=S$ has at least one solution in $\mathcal{D}^*$. (The functional αT is defined as usual by the relation $\langle\alpha T,\varphi\rangle=\langle T,\alpha\varphi\rangle$.)

E. Hewitt.

See also: Buck, p. 140.

Banach Spaces, Banach Algebras

Have

★**Banach, Stefan. Théorie des opérations linéaires.** Chelsea Publishing Co., New York, 1955. vii+254 pp. \$3.95.

Reprint by photo-offset of the original [Warsaw, 1932].

Have

★**Riesz, Frigyes, and Sz.-Nagy, Béla. Functional analysis.** Translated by Leo F. Boron. Frederick Ungar Publishing Co., New York, 1955. xii+468 pp. \$10.00.

Translation of the authors' Leçons d'analyse fonctionnelle, 2d ed. [Akadémiai Kiadó, Budapest, 1953; MR 15, 132].

Nikol'skiĭ, V. N. Some questions of best approximation in a function space. Uč. Zap. Kalininsk. Pedagog. Inst. 16 (1953), 119–160 (1954). (Russian)

The author studies the question of best approximation of elements of a space E of type F [for terminology see

Banach, Théorie des opérations linéaires, Warsaw, 1932, ch. 3] by means of linear combinations of a given denumerable sequence $\{u_k\}\subset E$. In connection with this there are introduced various types of metrics of the space E (T-metric, K-metric, canonical metric, weak T-metric, etc.).

Definition. Let $\{u_k\}$ be a basis for E. If for each $x\in E$ and each n the linear combination $\{u_k\}_1^n$ deviating least from x is unique and coincides with the nth partial sum of the expansion of x in the basis $\{u_k\}$, then the metric of E is called a T-metric with respect to $\{u_k\}$.

In each space E of type F one may introduce a T-metric with respect to an arbitrary basis which is equivalent to the original one (convergence in one implies convergence in the other). If (here the system $\{u_k\}$ need not be a basis) for each element of the form $x=\sum_{k=1}^m c_ku_k$ and each $n<m$ the linear combination of $\{u_k\}_1^n$ deviating least from x is unique and equal to $\sum_{k=1}^n c_ku_k$, then the metric of the space is called a weak T-metric with respect to $\{u_k\}$. Theorem: In order that a system $\{u_k\}$ be a basis for E it is necessary and sufficient that it be complete and that it be possible to introduce in E an equivalent weak T-metric.

A proof of the following proposition serves as an example of an application of this theorem: If

$$\lambda_{k+1}=\lambda_k(2p_k+1)\quad (k=1,2,\cdots;\ \lambda_k \text{ and } p_k \text{ integers}),$$

then the system of Čebyšev polynomials $\{T_{\lambda_k}(x)\}$ $(-1\leq x<1)$ is a basis (in the metric $C([-1,+1])$) of its closed linear hull.

There are also established a series of theorems similar to the one above. In particular, the following theorem concerning Banach spaces is proved: Let $\{u_k\}$ be a basis in the Banach space E. If $R_n(x)$ is nth remainder in the expansion of x in $\{u_k\}$ and $E_n(x)$ is the best approximation of x by linear combinations of $u_1,\cdots,u_n$, then

$$\mu\|R_n(x)\|\leq E_n(x)\leq\|R_n(x)\|,$$

where $\mu>0$ depends only on E and $\{u_k\}$.

Finally there are given necessary and sufficient conditions in order that the system $\{u_k\}$ be a basis in a Banach space E. *I. P. Natanson* (RŽMat **1955**, no. 5129).

Altman, M. A generalization of Newton's method. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 189–193.

L. V. Kantorovič [Trudy Mat. Inst. Steklov. **28** (1949), 104–144; MR **12**, 419] and others (cf. review of I. Fenyő [Acta Math. Acad. Sci. Hungar. **5** (1954), 85–93; MR **15**, 964] for other references) have considered the "modified Newton process" defined by

$$x_n=x_{n-1}-[P'(x_0)]^{-1}P(x_{n-1})\quad (n=1,2,\cdots),$$

to obtain solutions of the equation $P(x)=0$ near x_0. Here P is a mapping from a Banach space $\mathfrak{X}$ to another such space $\mathfrak{Y}$, and P' is its Fréchet derivative. The present author treats the case where $P'(x_0)$ maps $\mathfrak{X}$ onto all of $\mathfrak{Y}$, but is not assumed to be one-to-one, by considering the quotient space of $\mathfrak{X}$ modulo the kernel of $P'(x_0)$. Three theorems are given corresponding to the assumptions (1) $\|P'(x)-P'(x_0)\|<C$ for x in a sphere $S(x_0,r)$; (2) P' satisfies a Lipschitz condition in $S(x_0,r)$; and (3) P'' exists and is bounded on this sphere. *R. G. Bartle.*

Bartle, Robert G. Newton's method in Banach spaces. Proc. Amer. Math. Soc. **6** (1955), 827–831.

A generalization of the Newton process is obtained. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be (real or complex) Banach spaces, and consider a mapping $f:\mathfrak{G}\to\mathfrak{Y}$ where $\mathfrak{G}$ is an open set in $\mathfrak{X}$ and f has a continuous Fréchet derivative in $\mathfrak{G}$. Then the equation $f(x)=0$ may be solved by an iterative process:

$$x_{n+1}=x_n-[f'(z_n)]^{-1}f(x_n)\quad (n=0,1,\cdots),$$

provided that the initial approximation x_0 is such that $\|f(x_0)\|<\beta/2\lambda$ and the arbitrarily selected points z_n are such that $\|z_n-x_0\|<\beta$, where λ is an upper bound for $\|[f'(x_0)]^{-1}\|$ and β is a particularly selected modulus of continuity of f' at x_0. The solution is unique in the set $\|x-x_0\|\leq\beta$, and the rapidity of convergence is given in terms of β. The result is closely related to results of Kantorovič [Dokl. Akad. Nauk SSSR (N.S.) **59** (1948), 1237–1240; Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; Trudy Mat. Inst. Steklov. **28** (1949), 104–144; MR **9**, 537; **10**, 380; **12**, 419] and Mysovskih [ibid. **28** (1949), 145–147; Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 565–568; MR **12**, 419; **11**, 601]. *J. Cronin.*

Lyubič, Yu. I. On the belonging of the powers of an operator on a given vector to a certain linear class. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 881–884. (Russian)

An operator form of the Esclangon-type theorems is given, viz: Let T be a linear operator in a linear space S, $\mathfrak{M}$ a linear subspace of S, $\mathfrak{D}_n$ the domain of T^n. Under what circumstances is the implication "$x\in\mathfrak{D}_n\cap\mathfrak{M}$, $T^nx\in\mathfrak{M}\Rightarrow T^kx\in\mathfrak{M}$ $(k=1,2,\cdots,n-1)$" valid? Theorem: Let $\lambda_1,\lambda_2,\cdots,\lambda_n$ be n different numbers such that (a) $(T-\lambda_kI)x=\varphi$ $(k=1,\cdots,n)$ has a solution $x\in\mathfrak{M}$ for each $\varphi\in\mathfrak{M}$. (b) $P(T)x=0\Rightarrow x=0$ where $P(T)=\prod_{k=1}^n(T-\lambda_kI)$. Then, if $Q(T)$ is a polynomial operator of degree n, the implication

$$\text{"}x\in\mathfrak{D}_n\cap\mathfrak{M} \text{ and } Q(T)x\in\mathfrak{M}\Rightarrow T^kx\in\mathfrak{M}\ (k=1,\cdots,n-1)\text{"}$$

is valid. The latter half of the paper is devoted to applications of this result, where $T=d/dt$, and to estimates for $\|T^kx\|$ in terms of $\|T^n(x)\|$ $(k=1,\cdots,n-1)$. *B. Gelbaum.*

Henstock, R. Linear functions with domain a real countably infinite dimensional space. Proc. London Math. Soc. (3) **5** (1955), 238–256.

The results obtained in this paper are covered by the following remarks. Let m be the Banach space (of dimension c) of bounded real sequences $\xi=\{x_n\}$, with $\|\xi\|=\sup_n|x_n|$. Let Q be the set of all ξ with $\|\xi\|\leq\frac{1}{2}$. Regard Q as the countable product of the interval $[-\frac{1}{2},\frac{1}{2}]$ by itself, and endow it with the usual product measure. The author is concerned with linear functionals J which are measurable on Q but possibly discontinuous. The "zero or one" principle yields: if E is a linear subspace of m, then $E\cap Q$ either has measure 0 or measure 1. Theorem 1: If J is finite-valued and measurable almost everywhere (a.e.) on Q, then there is a sequence $\{c_n\}$ such that $\sum c_n^2$ converges, and $J(\xi)=\sum c_nx_n$ for a.e. ξ in Q. Conversely, if $\sum c_n^2$ converges, then $\sum c_nx_n$ converges for a.e. ξ and defines a measurable functional J. The author also considers the behavior of a family $\{J_t\}$ of measurable linear functionals, and discusses the existence of $\lim J_t(\xi)$ for a.e. ξ in Q, as t increases (avoiding a set of t of zero measure which may depend upon ξ). In the last section, the author turns to a discussion of the Borel property of a general Toeplitz sequence transformation, and obtains an alternate form of the conditions given by J. D. Hill [Pacific J. Math. **1** (1951), 399–409; MR **13**, 340]. [The reviewer would like to point out that Theorem 1 is very closely connected with a classical theorem of Kolmogorov,

dealing with series of random variables [e.g., Halmos, Measure theory, Van Nostrand, New York, 1950, p. 197; MR 11, 504]. In connection with the Borel property, see also J. D. Hill, Pacific J. Math. 4 (1954), 227–242; MR 15, 950; Tsuchikura, Proc. Japan Acad. 27 (1951), 141–145; MR 13, 739; G. G. Lorentz, Duke Math. J. 22 (1955), 129–141; MR 17, 147.] *R. C. Buck* (Madison, Wis.).

Bohnenblust, H. F., and Karlin, S. Geometrical properties of the unit sphere of Banach algebras. Ann. of Math. (2) 62 (1955), 217–229.

The authors are concerned with relations between geometric properties (vertices, extreme points, tangents) of admissible unit spheres in a Banach algebra A and algebraic properties (identity, ideals, involutions of A which follow from the usual assumptions in a Banach algebra that $\|u\|=1$ and $\|ab\|\leq\|a\|\ \|b\|$. They show that the unit u of a Banach algebra is always a vertex of the unit sphere S (that is, the hyperplanes of support at u intersect in a set containing no point but u). Now suppose that A is a commutative (and *-) algebra. Then the inf of (*-) admissible norms in A is the spectral radius function. Also, if P is the set of all f in A^* for which $f(x^*x)\geq 0$ for all x in A, and if Γ is the w^*-closed convex hull in A^* of the set of all maximal ideals of A, then A is symmetric if and only if $P=\Gamma$. *M. M. Day* (Urbana, Ill.).

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. On kernel functions. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 14–19.

Axioms for a ring R with an involution and an R-valued inner product are given. These are a mutation of previously published axioms [same Rep. 4 (1954), 299–308; MR 16, 597]. Some simple consequences are drawn and examples from the theory of functions of 2 complex variables are indicated. *E. Hewitt* (Princeton, N.J.).

Arens, Richard, and Calderón, A. P. Analytic functions of several Banach algebra elements. Ann. of Math. (2) 62 (1955), 204–216.

Let A denote a commutative Banach algebra with unit over the complex field C, and let $\mathfrak{M}$ denote the space (in the weak topology) of maximal ideals of A. If $M\in\mathfrak{M}$, and $a\in A$, let $M(a)$ denote the image of a under the natural homomorphism of A onto C whose kernel is M. If $F=\{a_1, \cdots, a_n\}$ is a finite subset of A, then let $f_A(M)=\{(M(a_1), \cdots, M(a_n)):M\in\mathfrak{M}\}$. The authors show that if $\mathfrak{F}$ is any function of n complex variables analytic on a neighborhood of $f_A(\mathfrak{M})$, then there is an $a\in A$ such that $M(a)=\mathfrak{F}((M(a_1), \cdots, M(a_n))$ for all $M\in\mathfrak{M}$. Šilov had obtained the result previously under the additional assumption that A is generated by F [Mat. Sb. N.S. 32(74) (1953), 353–364; MR 14, 884]. This theorem is applied to give a simple proof of the main theorem of Šilov's paper; namely that if $\mathfrak{M}$ is disconnected, then A is the direct sum of two closed ideals.

The authors then turn to a study of multiple-valued analytic functions and multiple power series on A. A sample theorem of the kind obtained is the following: Let h be a complex-valued continuous function on $\mathfrak{M}$, and let G be any complex-valued function regular on a neighborhood of $S=\{(h(M), M(a_1), \cdots, M(a_n))\}$ such that G vanishes on S, while the partial derivative G_1 never vanishes on S. Then, there is an $a_0\in A$ such that $h(M)=M(a_0)$ for all $M\in\mathfrak{M}$. *M. Henriksen.*

Bauer, Heinz. Reguläre und singuläre Abbildungen eines distributiven Verbandes in einen vollständigen Vektorverband, welche der Funktionalgleichung $f(x\vee y)+f(x\wedge y)=f(x)+f(y)$ genügen. J. Reine Angew. Math. 194 (1955), 141–179.

This paper continues the theme of a previous communication by the same author [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 89–117 (1954); MR 16, 50]. For terminology and notation, see the review cited. The first half of the present paper, mostly expository, summarizes the present state of the theory of valuations of a lattice V with values in a complete vector lattice W. The second half deals with various decompositions of valuations, the main theorems being based on a theorem of F. Riesz [Ann. of Math. (2) 41 (1940), 174–206; MR 1, 147]. Let S be a subset of V. For $f\in\Phi_e$ and $f\geq 0$, let $f_S=\inf\{h:h\in\Phi_e, h\geq 0, h(x)=f(x) \text{ for } x\in S\}$. For $f\in\Phi_e$, let $f_S=(f^+)_S-(f^-)_S$. A number of formal properties of f_S are established. An element f of Φ_e is S-regular (S-singular) if $f_S=f$ ($f_S=0$). The decomposition theorem asserts that Φ_e admits a direct order-preserving decomposition into the band of S-regular elements and the band of S-singular elements. This decomposition is given by $f=f_S+(f-f_S)$. This theorem generalizes a theorem of Hahn and Rosenthal [Set functions, Univ. of New Mexico Press, 1948, p. 42; MR 9, 504]. An element $f\in\Phi_e$ is said to be g-continuous (where $g\in\Phi_e$) if f lies in the smallest band in Φ_e that contains g. If $|f|\wedge|g|=0$, then f is said to be purely g-discontinuous. Then Φ_e admits a direct order-preserving decomposition into the band of g-continuous elements and the band of purely g-discontinuous elements. This theorem generalizes Lebesgue's decomposition for set-functions. *E. Hewitt* (Princeton, N.J.).

Cristescu, Romulus. Espaces partiellement ordonnés pseudo-normés. Com. Acad. R. P. Romîne 4 (1954), 15–20. (Romanian. Russian and French summaries)

This paper covers much the same ground as another communication by the same author [Acad. Repub. Pop. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. 6 (1954), 297–304; MR 16, 715]. *E. Hewitt* (Princeton, N.J.).

Cristescu, Romulus. La notion d'intégrale dans les espaces semi-ordonnés. Com. Acad. R. P. Romîne 2 (1952), 205–208. (Romanian. Russian and French summaries)

This paper is very like another by the same author [Acad. Repub. Pop. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. 4 (1952), 291–310; MR 15, 721]. *E. Hewitt.*

Cristescu, Romulus. Un théorème sur la représentation des opérations linéaires. Com. Acad. R. P. Romîne 5 (1955), 655–659. (Romanian. Russian and French summaries)

Let T be a set with a σ-algebra of subsets $\mathfrak{T}$ and a countably additive real-valued measure μ on $\mathfrak{T}$ such that $\mu(T)$ is finite. Let S_T be the space of all measurable real-valued functions on T that are finite almost everywhere (μ). For certain subspaces $\mathfrak{X}$ of S_T and certain linear mappings U of $\mathfrak{X}$ into certain lattice-ordered vector spaces $\mathfrak{Y}$, the author gives a representation theorem

$$U(f)=\int_T f(t)dm^*(t),$$

where m^* is a countably additive $\mathfrak{Y}$-valued measure on $\mathfrak{T}$ that is absolutely continuous with respect to μ. *E. Hewitt.*

See also: Gagua, p. 194.

Hilbert Space

Edwards, D. A. Vector-valued measure and bounded variation in Hilbert space. Math. Scand. 3 (1955), 90–96.

There exists a strongly continuous function z from the closed unit interval into a (real) Hilbert space X such that if $f_x(t)=(z(t), x)$, then f_x is of bounded variation on the interval for every x in X, but if $g(s, t)=(z(s), z(t))$, then g is not of bounded variation (in the sense of Vitali) on the square. *P. R. Halmos* (Chicago, Ill.).

Calderón, Alberto, et Devinatz, Allen. Sur certaines courbes dans l'espace de Hilbert. C. R. Acad. Sci. Paris 241 (1955), 539–541.

Calderón, Alberto-P., et Devinatz, Allen. Sur certaines courbes à courbure constante dans l'espace de Hilbert. C. R. Acad. Sci. Paris 241 (1955), 586–587.

Let $f(t)$ be a parametrised curve in Hilbert space such that $\|f^{(n)}(t)\|$ is independent of t for all $n\geqq 0$. Then if the numbers μ_n defined by $\mu_{2n}=\|f^{(n)}\|^2$, $\mu_{2n+1}=0$ can be represented uniquely as the moments of a mass distribution, $f(t)$ has the form $f(t)=U_t x$, where U_t is a one-parameter group of unitary transformations. If the condition on theconstants μ_n is not satisfied, an example is given which shows that the conclusion is no longer valid. *J. Schwartz* (New York, N.Y.).

Raimi, Ralph A. Compact transformations and the k-topology in Hilbert space. Proc. Amer. Math. Soc. 6 (1955), 643–646.

L'auteur prouve que, dans un espace de Hilbert H, la topologie de la convergence compacte (sur H considéré comme son propre dual) est identique à la topologie la moins fine rendant continue toutes les applications linéaires compactes de H dans lui-même. Le point essentiel consiste à prouver que, si S est la boule unité dans H, K une partie compacte quelconque de H, il existe une application compacte u telle que $K\subset u(S)$; la méthode utilisée ne semble pas susceptible de s'étendre à des espaces de Banach plus généraux. *J. Dieudonné* (Evanston, Ill.).

Nieminen, Toivo. On the spectral theorems of unitary and selfadjoint transformations. Ann. Acad. Sci. Fenn. Ser. A. I. no. 187 (1955), 37 pp.

The author presents another proof of the spectral theorem using some complex-variable theorems of Nevanlinna. *F. H. Brownell* (Seattle, Wash.).

Devinatz, A., Nussbaum, A. E., and von Neumann, J. On the permutability of self-adjoint operators. Ann. of Math. (2) 62 (1955), 199–203.

Theorem 1. If the product of two unbounded self-adjoint operators T_1, T_2 is an extension of a third self-adjoint operator T, then every projection in the range of the resolution of the identity of T_1 commutes with every projection in the range of the resolution of the identity of T_2. Corollary 1. Under the hypotheses of Theorem 1, $T=T_1T_2$. *J. Schwartz* (New York, N.Y.).

Kadison, R. V. Isomorphisms of factors of infinite type. Canad. J. Math. 7 (1955), 322–327.

The implementation of *-isomorphisms of rings of operators has been studied by Pallu de La Barrière [Bull. Soc. Math. France 82 (1954), 1–52; MR 16, 491], Misonou [Proc. Japan. Acad. 29 (1953), 482–485; MR 16, 264], and Griffin [Trans. Amer. Math. Soc. 75 (1953), 471–504; 79 (1955), 389–400; MR 15, 539; 17, 66]. Their work, however, omits the cases in which the *-isomorphic rings are of type II_∞ with type II_1 commutants [in the terminology of Murray and von Neumann, Ann. of Math. (2) 37 (1936), 116–229]. The present paper closes this gap by the introduction of an invariant termed the "linking operator": If ϕ is a *-isomorphism between two rings of operators M_1 and M_2 of the type mentioned above acting upon Hilbert spaces H_1 and H_2 respectively, we shall call the operator $D(\phi(E))$ in the center of M_2, the linking operator for ϕ, where E is the projection in M_1 with range $[M_1'x]$, where x is a unit vector such that $[M_1x]=H_1$, and where D is the center-valued dimension function on M_2 normalized so that $D(F)=I$. (F defined for M_2 in the same way as E is defined for M_1; assuming M_1, M_2 have countably decomposable centers.) Using this idea, the author proves that ϕ can be implemented by a unitary operator if and only if the linking operator is the identity operator. An application of this result in turn to automorphisms is used to show the existence of non-unitarily induced automorphisms. *E. L. Griffin, Jr.*

Borisovič, Yu. G. On an estimate of the number of critical points of functionals. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 205–207; errata 103 (1955), 952. (Russian)

After finding some new properties of Krasnosel'skiĭ's concept of genus (Russian: "rod") of a set [for definition see the review, MR 14, 55, of Krasnosel'skiĭ's paper in Uspehi Mat. Nauk (N.S.) 7 (1952), no. 2(48), 157–164; where this reviewer translated "rod" as "kind"], the author proves a number of theorems on Lagrange multipliers for conditioned extrema of weakly continuous functionals f on real separable Hilbert space H, for example the following. If Γ is the gradient operator of a weakly continuous, even, positive functional f uniformly differentiable on each sphere of H, with $f(x)=0$ and $\Gamma(x)=0$ only at $x=0$, and if $f(x)\to\infty$ when $\|x\|\to\infty$; if L is a linear self-adjoint operator in H with n positive eigenvalues counting multiplicities, then on each surface $(Lx, x)=c>0$ there are at least $2n$ critical points x_i, in which $\Gamma x_i-\lambda_i x_i=0$. *J. M. Danskin* (Princeton, N.J.).

Friedman, Bernard. An abstract formulation of the method of separation of variables. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. BR-12 (1955), i+24 pp.

This paper concerns the inversion of an operator defined on the direct product of two Hilbert spaces as a finite sum of products of certain operators defined on the two factor spaces. There is considerable overlap with the work of Cordes [Math. Ann. 125 (1953), 401–434; 128 (1954), 257–289; 373–411; Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1954, 51—69; MR 14, 1096; 16, 597; 15, 967]. The author's results are contained in two theorems, which are proved only for pure point spectra, and whose formulation in the more general case is not adequately precise. A number of applied examples of partial differential operators are discussed. *F. H. Brownell* (Seattle, Wash.).

See also: Riesz and Sz.-Nagy, p. 175; Slobodyanskiĭ, p. 195; Mackina, p. 180.

TOPOLOGY

Hönig, Chaim Samuel. Sur les topologies semi-régulières. An. Acad. Brasil. Ci. **27** (1955), 1–6.

Let E be a set and T a topology on E. Let A be a family of subsets of E closed under the formation of finite unions and arbitrary intersections, every element of which has void interior under the topology T. Let T_A be the topology on E obtained by adjoining A to the family of sets closed under T. Let $\{T\}$ denote the family of all such topologies T_A on E. The author announces without proof a number of theorems, some about topologies T_A and others proved by their use. He points out that a number of his results are either not new or simple extensions of known facts. The following are typical of his new results. 1. A minimal Hausdorff topology is semi-regular. (That is, open sets G for which $G=G^{-\prime-\prime}$ are a basis for open sets.) 2. If T is semi-regular, then $\{T\}$ is the class of all topologies on E for which the associated semi-regular topology is T. 3. A Hausdorff topology is absolutely closed if and only if the associated semi-regular topology is minimal and Hausdorff. 4. A topology is extremally disconnected (that is, it is Hausdorff and the closure of every open set is open) if and only if it has the form T_A, where T is extremally disconnected and completely regular. (Reviewer's note. The author has changed the reviewer's original definition of extremal disconnectivity [Duke Math. J. **10** (1943), 309–333; MR **5**, 46].) 5. For an arbitrary T, the maximal elements of $\{T\}$ admit no pairs of complementary dense subsets. *E. Hewitt* (Princeton, N.J.).

Banaschewski, Bernhard. Über den Ultrafilterraum Math. Nachr. **13** (1955), 273–281.

The author studies the Stone-Čech compactification of an infinite discrete space E as the space Ω of all ultrafilters on E (suitably topologized). In the first part of the paper, he gives an alternate proof of Pospíšil's theorem [Ann. of Math. (2) **38** (1937), 845–846] that there are exp exp $|E|$ distinct ultrafilters on E, and obtains other theorems on cardinal numbers of sets of ultrafilters of prescribed types. In the remainder of the paper, the group of homeomorphisms of Ω onto itself is studied. Most of these theorems seem to be generally known, but, to the reviewer's knowledge, a systematic presentation of them is given here for the first time. *M. Henriksen* (Lafayette, Ind.)

Richter, Wolfgang. Zur Theorie der Berührungsräume. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 3 (1954), 227–233.

Different results are given concerning proximity structures, topologies, and algebras of complex-valued functions on a given set E. Most of them are essentially known (often in a different form); e.g. the theorem 9 concerning one-to-one correspondence between proximity structures on E and compact extensions of E is essentially due to Yu. Smirnov [Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 895–898; MR **14**, 1107].

The following result, which seems to be stated (and proved) for the first time in the present note, may be formulated as follows: if E is a proximity space, then the set $\mathfrak{A}_\delta$ of all δ-continuous functions on E is a Gelfand algebra; if $\mathfrak{A}$ is a Gelfand algebra on a set E, then there exists exactly one proximity structure δ on E such that $\mathfrak{A}_\delta=\mathfrak{A}$ (in the author's terminology, a Gelfand algebra of E is a complete subalgebra $\mathfrak{A}$ of the normed algebra of all bounded complex-valued functions on E such that $1 \in \mathfrak{A}$ and $\mathfrak{A}$ contains with every f its conjugate f^*). *M. Katětov* (Prague).

Nagata, Jun-iti. On complete metric space. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 47–53.

Let R be a complete metric space, with metric d. A uniform neighborhood (in the extended sense) is defined as a real-valued function f on R such that I) $f(x)\geqq\varepsilon$ for some $\varepsilon>0$ and all $x \in R$; II) if $f(x)\geqq 1/2n$, and $d(x, y)\leqq 1/n$, then $f(y)\leqq 1/n$ ($n=1, 2, \cdots; x, y \in R$). Under the ordinary partial ordering of functions, the set of all such uniform neighborhoods forms a lattice. Theorem: this lattice characterizes the uniform topology of R. Other results are obtained as well. *E. Hewitt* (Princeton, N.J.).

Lee, Hsiao-chuan. On completion of metric spaces. Acta Math. Sinica 3 (1953), 154–165. (Chinese. English summary)

Mainly expository. *Ky Fan* (Notre Dame, Ind.).

Morita, Kiiti. A condition for the metrizability of topological spaces and for n-dimensionality. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1955), 33–36.

The two main results of this paper are the following. Theorem 1. In order that a T_1 space X be metrizable it is necessary and sufficient that there exist a countable collection $\{\mathcal{F}_i\}$ of locally finite closed coverings of X satisfying: (1) For any neighborhood U of any x in X there is an i such that $\bigcup\{F : x \in F \in \mathcal{F}_i\}\subset U$. Theorem 2. In order that a T_1 space X be metrizable and of (Lebesgue covering) dim $\leqq n$ it is necessary and sufficient that there exist a countable collection $\{\mathcal{F}_i\}$ of locally finite closed coverings satisfying (1) of Theorem 1 and a non-empty set Ω such that:

(2) $\quad \mathcal{F}_i=\{F(\alpha_1, \cdots, \alpha_i):\alpha_\nu \in \Omega, \nu=1, \cdots, i\}$,

where $F(\alpha_1, \cdots, \alpha_i)$ may be empty;

(3) $\quad F(\alpha_1, \cdots, \alpha_{i-1})=\bigcup\{F(\alpha_1, \cdots, \alpha_{i-1}, \gamma):\gamma \in \Omega\}$;

and (4) The order of $\mathcal{F}_i$ does not exceed $n+1$ for each i.

Several critical points of the proofs are dependent on results in another paper of the author [Math. Ann. **128** (1954), 350–362; MR **16**, 501]. *Haskell Cohen.*

Hashimoto, Hiroshi. On the resemblance of point sets. Math. Japon. 3 (1954), 53–56.

For any subset X of a topological space, let $D(X)$ denote the set of points at which X is not of the first category. The set $X+D(X)$ is called D-closure of X, and simple consequences of this definition are stated (proofs are obvious); e.g., X is a Baire set if and only if it is a difference of two D-closed sets (this fact is referred to as "resemblance" between Baire sets and sets which are differences of two closed sets). *M. Katětov.*

Wallace, A. D. Struct ideals. Proc. Amer. Math. Soc. **6** (1955), 634–638.

Let X denote a Hausdorff space, and L a subset of $X\times X$ (i.e., a relation on X). A non-empty subset A of X is called an L-ideal if $(X\times A)\cap L\subset A\times X$. If L is closed, non-empty, and transitive, then L is called a struct on X. Theorems are proved on L-ideals, for the most part under the hypothesis that L is a struct on a compact space, that generalize corresponding results on topological

semigroups. A sample theorem is that for any proper L-ideal A, there exists a maximal [minimal] L-ideal containing [contained in] A, which is open [closed] [see Koch and Wallace, Duke Math. J. **21** (1954), 681–686; MR **16**, 112]. *M. Henriksen* (Lafayette, Ind.).

Mackina, R. Yu. On one-to-one continuous images of Hilbert space. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 267–272. (Russian)

Let H and H^* be separable real Hilbert spaces. Let E be a linear Borel set of class $\alpha \geqq 2$, that is a continuous one-to-one image of a Baire space. Then there exists a one-to-one continuous mapping φ of H into H^* such that $\varphi(H)$ is a Borel set of class α and such that $\varphi(H)$ contains a closed set that is homeomorphic to E. [See also the earlier communication of the same author, same journal **15** (1951), 533–544; MR **13**, 573.] *E. Hewitt.*

Taĭmanov, A. D. On universal sets. Mat. Sb. N.S. **37** **(79)** (1955), 117–120. (Russian)

Let E_k denote k-dimensional real Euclidean space, with points $x=(x_1, x_2, \cdots, x_k)$ $(k=1, 2, 3, \cdots)$. Let n be a positive integer, and let $\mathcal{E}$ denote an arbitrary locally compact set in the hyperplane $x_1=0$ in E_{n+1}. There exists a locally compact set $U \subset E_{n+1}$ such that for every $\mathcal{E}$, there exists a number a such that

$$\{(0, x_2, \cdots, x_{n+1}) : (a, x_2, \cdots, x_{n+1}) \in U\} = \mathcal{E}.$$

The author has previously given an analogous construction for plane sets contained in E_4 [Mat. Sb. N.S. **30(72)** (1952), 465–482; MR **14**, 191]. *E. Hewitt.*

Ward, L. E., Jr. A note on dendrites and trees. Proc. Amer. Math. Soc. **5** (1954), 992–994.

A characterization of dendrites and trees (a tree is a compact connected Hausdorff space in which every two points are separated by a third point) in terms of partial order is given [the note leans upon a previous paper of the author, same Proc. **5** (1954), 144–161; MR **16**, 59]. Main result: a compact Hausdorff space X is a tree if and only if it admits of a semicontinuous order-dense partial order such that, for any $x, y \in X$, $\{z | z \leqq x, z \leqq y\}$ is a non-void chain and every $\{z | x \leqq z, z \neq x\}$ is open. *M. Katětov* (Prague).

Burgess, C. E. Collections and sequences of continua in the plane. Pacific J. Math. **5** (1955), 325–333.

The author proves several theorems of which the following is typical and perhaps the most interesting. If, in the plane, W is a collection consisting of n mutually exclusive bounded continua and G is a collection consisting of n^2-n+1 mutually exclusive continua each intersecting every continuum of W, then some continuum of G contains a bounded continuum which intersects every continuum of W. *R. D. Anderson* (Princeton, N.J.).

Jones, F. Burton. On a certain type of homogeneous plane continuum. Proc. Amer. Math. Soc. **6** (1955), 735–740.

The author [Bull. Amer. Math. Soc. **60** (1954), 560] and R. H. Bing [ibid. **60** (1954), 559] showed independently that there exists a homogeneous, decomposable, bounded plane continuum other than a simple closed curve. In this paper the author investigates homogeneous, decomposable, compact, metric continua and obtains the following result: If M is such a continuum, there exists a non-degenerate continuous collection G of mutually exclusive, homogeneous continua (each two of which are homeomorphic) filling up M such that G with respect to its elements as points is a homogeneous, aposyndetic continuum. If, in addition, M lies in the plane, then each element of G is an indecomposable continuum which does not separate the plane and G (with respect to its elements) is a simple closed curve. For each point x of M, the element of G containing x is the set consisting of x together with all points y of M such that M is not aposyndetic at y with respect to x. In the example mentioned above the elements of G are pseudo-arcs. *E. Dyer* (Baltimore, Md.).

Wada, Hidekazu. Local connectivity of mapping spaces. Duke Math. J. **22** (1955), 419–425.

The author proves: If X is an m-dimensional finite polytope, and Y is Hausdorff and SLC^n (i.e. for every $y \in Y$ and neighborhood U of y there is a neighborhood V, $y \in V \subset U$ such that every p-sphere, $0 \leqq p \leqq n$, in V is contractible in V) then the function space Y^X with the compact-open topology is SLC^{n-m}. Reviewer's remark: for Y first countable and LC^n (in the usual sense) and X a compact metric space, the result that Y^X is LC^{n-m} follows very simply from an argument of Kuratowski [Fund. Math. **24** (1935), 269–287, pp. 271, 285]. *J. Dugundji* (Los Angeles, Calif.).

Krasnosel'skiĭ, M. A. On computation of the rotation of a vector field on the n-dimensional sphere. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 401–404. (Russian)

Let U be a periodic homeomorphism of the Euclidean n-sphere S^n onto itself with period p, so that $U^px=x$ for $x \in S^n$. Let γ_U be the degree of the mapping U. Let V be a periodic homeomorphism of Euclidean $(n+1)$-space R^{n+1} onto itself, the degree of which is γ_V and the period of which is q, with p divisible by q and $\|Vx\|=\|x\|$ for $x \in R^{n+1}$. The author's main result is as follows: Suppose none of the mappings $U^1, \cdots, U^{p-1}$ has a fixed point. Let Φ and Ψ be continuous vector fields without null vectors on S^n, satisfying the conditions

$$\Phi Ux = V\Phi x, \; \Psi Ux = V\Psi x \quad (x \in S^n).$$

Then $\gamma_\Phi=\gamma_\Psi$ if $\gamma_U\gamma_V=-1$, and $\gamma_\Phi=\gamma_\Psi \pmod p$ if $\gamma_U\gamma_V=1$. *J. M. Danskin* (Princeton, N.J.).

Hedlund, G. A. A class of transformations of the plane. Proc. Cambridge Philos. Soc. **51** (1955), 554–564.

All transformations φ of the complex plane of the form $\varphi(re^{i\theta})=re^{f(\theta)}e^{i(\theta+\beta)}$, where β/π is irrational and $f(\theta)$ belongs to the class P of real continuous functions with period 2π, are shown to be of three types: either (1) φ is topologically equivalent to a stretching $\varphi(z)=sz$, $0<s\neq 1$, or (2) to the rotation $\varphi(z)=ze^{i\beta}$, or else (3) φ is regionally transitive. Let $a=\int_0^{2\pi} f$ and $b=\sup \sum_{p=1}^n f(\theta+p\beta)$. Criteria for the three cases are: (1) $a\neq 0$, (2) $a=0$, $b<\infty$, (3) $a=0$, $b=\infty$. A key result is that if the positive or negative semi-orbit of some point is contained in an annulus $r_1<|z|<r_2$ then every orbit closure is a simple closed curve, $f(\theta)=g(\theta+\beta)-g(\theta)$ for some g in P, and φ is of type (2). It follows, in answer to a question raised by Besicovitch [same Proc. **47** (1951), 38–45; MR **12**, 519], that in the transitive case no orbit can be bounded away from both 0 and ∞. Conditions on the Fourier coefficients of f sufficient to insure that φ is of type (2) or (3) are given. By the correspondence $re^{i\theta} \to (\theta, \log r)$ the set $0<|z|<\infty$ is mapped onto a cylinder C, and φ corresponds to an area-preserving transformation of C which is either transitive or topologically equivalent to a rotation or translation of C. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Weier, Josef. Sur les points de ramification d'une déformation finie. J. Math. Pures Appl. (9) 34 (1955), 137–143.

If $F:X\times I\to X$, set $F^t=F|X\times t$ and $S=\{x, t|F(x, t)=x\}$. The author assumes X is a polyhedron, S a finite 1-dimensional homogeneous polyhedron, and each F^t has a finite number of fixed points. $(x_0, t_0)\in S$ is a 'point of ramification' if for any open $U\ni x$ and $\varepsilon>0$, $|t-t_0|<\varepsilon\Rightarrow F^t$ has at least two distinct essential fixed points in U. S is given the triangulation without vertices of degree two. The behavior of t on each edge of S turns S into a directed graph and assigns to each edge a left and a right vertex. For all (x, t) in the interior of an edge the index of x under F^t is independent of (x, t) and is called the index, n, of the edge. The edges of S are called 'singularities'' and, if $n\neq 0$, "essential singularities." The key observation is that if (x, t) is a vertex of S, $t\neq 0, 1$, then the index of x under F^t is equal to the sum of the indices of the edges of which (x, t) is a right (left) vertex. Typical of the five theorems is: A necessary and sufficient condition that a vertex of S is a point of ramification is that it is the left (or the right) vertex of at least two essential singularities. *S. Stein* (Davis, Calif.).

Burks, Arthur W., McNaughton, Robert, Pollmar, Carl H., Warren, Don W., and Wright, Jesse B. The folded tree. I, II. J. Franklin Inst. 260 (1955), 9–24, 115–126.

An n-bay folded tree is a diagram composed of lines and vertices, i.e. a linear graph, in which each vertex has one connection designated as the input and two as outputs. Vertices are classified as lying in numbered bays with the outputs of vertices in any bay but the last connected to the inputs of two elements in the next bay. The first bay contains one vertex and by the connection rule the number of vertices appearing in each successive bay is doubled. Each vertex is labeled with one of n labels which are so arranged that, by following any route through successive bays from the first to the last, passage is made through just one vertex with each of the n labels.

Such diagrams are useful in application to switching circuits and are the same as the disjunctive trees of C. E. Shannon, who gives a less complete treatment. [Bell System Tech. J. 28 (1949), 59–98; MR 10, 671]. Complete decoding of n independent signals is achieved by an n-bay folded tree if one thinks of the vertices as double-throw relay contacts or equivalent switching elements which are operated by one of n signals depending on the label. A sequence of numbers called the loading sequence is constructed in which ith number in the sequence is equal to the number of vertices in the tree having the ith label. Three conditions (called conditions of admissibility) on the loading sequence are shown to be necessary and sufficient for the realization of a folded tree having that sequence. The proof of sufficiency is constructive in that it provides a practical method for forming a folded tree with a given admissible loading sequence.

The number of vertices in an n-bay folded tree is shown to be minimal over a larger class of diagrams which also provide complete decoding. An extension of the definition of admissibility is given which is shown to be necessary for the realization of a generalized folded tree in which the vertices have m outputs instead of two.

D. E. Muller (Urbana Ill.).

See also: Fenchel, p. 185.

Algebraic Topology

★**Pontryagin, L. S. Gladkie mnogoobraziya i ih primeneniya v teorii gomotopii. [Smooth manifolds and their applications in homotopy theory.]** Trudy Mat. Inst. im. Steklov. no. 45. Izdat. Akad. Nauk SSSR, Moscow, 1955. 139 pp. 6.30 rubles.

The present volume consists of a detailed account of the methods introduced by the author and developed by him and by V. A. Rohlin to study the homotopy classification problem for maps of S^{n+k} in S^n [see L. S. Pontryagin, C. R. (Dokl.) Acad. Sci. URSS (N.S.) 19, 147–149 (1938); 70, 957–959 (1950); MR 13, 57; V. A. Rohlin, ibid. (N.S.) 80, 541–544; 81, 19–22 (1951); 84, 221–224 (1952); MR 13, 674; 14, 573].

Chapter I consists of a study of smooth (differentiable) manifolds M^k. This study is taken considerably further than is required for the subsequent applications. Of particular value is a comparatively simple account of the Whitney theorem on the smooth embeddability of M^k in E^{2k+1}, Euclidean space of $(2k+1)$ dimensions, and a discussion of the properties of the singular points of a smooth map of M^k in E^r, $r<2k+1$, particularly $r=2k$, $2k-1$.

In Chapter II, the role of smooth manifolds in the homotopy classification problem is explained. First it is proved that every map $f:S^{n+k}\to S^n$ may be approximated by a smooth map and that homotopies between smooth maps may be approximated by smooth homotopies. If $p\in S^n$ is a fixed base point, and if f is smooth, then $f^{-1}(p)$ is a smooth manifold M^k. Moreover, if $e_1, \cdots, e_n$ is a system of linearly independent vectors tangent to S^n at p, then f induces a system U of n linearly independent vectors $u_1(x), \cdots, u_n(x)$, normal to M^k, at each point x of M^k; moreover, the vectors $u_1(x), \cdots, u_n(x)$ depend continuously, indeed smoothly, on x. An object consisting of a smooth manifold M^k and a normal field U is called by Pontryagin a normally framed manifold (M^k, U) of S^{n+k} (or E^{n+k}). Then every normally framed manifold is the normally framed manifold of some map $f:S^{n+k}\to S^n$ and two maps are homotopic if and only if the associated normally framed manifolds are homologous; by definition, $(M_1^k, U_1)\sim(M_2^k, U_2)$ if and only if there exists a normally framed manifold (M^{k+1}, U) in $E^{n+k}\times I$ such that $\partial M^{k+1}=M_2^k-M_1^k$ and U induces U_i in M_i^k, $i=1, 2$. Thus the homotopy problem is replaced by a homology problem for normally framed manifolds.

Standard concepts associated with the homotopy problem are readily translated in terms of submanifolds of S^{n+k}. Chapter II ends with the interpretation of the Freudenthal suspension and the proof, in these terms, of the classical Freudenthal theorems. Chapter III opens with an account of the classification of mappings of an n-dimensional manifold in S^n and proceeds to the Hopf invariant. Consideration of the case of maps $S^3\to S^2$ (giving rise to one-dimensional submanifolds) leads to the definition of a mod 2 invariant for maps $S^{n+1}\to S^n$, $n>2$, and this is proved in Chapter IV uniquely to determine the homotopy class. A similar programme is carried out for maps $S^{n+2}\to S^n$. The author remarks that the classification problem for normally framed submanifolds (M^k, U), $k\leq 2$, presents no great difficulty but that the case $k=3$ is already formidable; he refers to Rohlin's solution of the problem in this case.

The recommendation which this admirably written book must carry is not as a description of a powerful

method for computing homotopy groups of spheres, but as an exposition of the author's 'infinitesimal' method and an introduction to the properties of smooth manifolds.

P. J. Hilton (Cambridge, England).

Noguchi, Hiroshi. On the problem of the invariance of homotopical stability of points under Cartesian multiplication. Proc. Amer. Math. Soc. 6 (1955), 651–655.

The author constructs an example which may be specialized to one of Begle [Bull. Amer. Math. Soc. 49 (1943), 386–387; MR 4, 225] and which answers in the negative the question of Borsuk and Jaworowski [Fund. Math. 39, 159–175 (1953); MR 15, 51] as to whether the homotopical stability of points is preserved under Cartesian multiplication.

Haskell Cohen.

Yokota, Ichiro. On the cell structure of the octanion projective plane II. J. Inst. Polytech. Osaka City Univ. Ser. A. 6 (1955), 31–37.

Using matric representation, the author proves that the projective plane of octaves is a CW-complex [see J. H. C. Whitehead, Bull. Amer. Math. Soc. 55 (1949), 213–245; MR 11, 48], in which a 16-cell is attached to the 8-sphere by the Hopf map. [Remark of the reviewer: Eventually this is the G. Hirsch construction of the plane of octaves; see Colloq. Internat. Centre Nat. Rech. Sci., Paris, no. 12 (1949), pp. 35–42; MR 11, 610.] Some homotopy groups are calculated; concerning other homotopy groups a conjecture is made.

H. Freudenthal (Utrecht).

Spanier, E. H., and Whitehead, J. H. C. Obstructions to compression. Quart. J. Math. Oxford Ser. (2) 6 (1955), 91–100.

A map $f:(X, A)\rightarrow(Y, B)$ is called compressible into $Y'\subset Y$ if it is deformable into a map of X into Y' keeping the image of A always in B. The authors define an obstruction to compression and give some of its fundamental properties. It is assumed throughout that: (1) X, Y are CW-complexes and A, B sub-complexes; (2) for each $n\geqq 0$ there are only finitely many n-cells in $Y-B$. First, a co-cohomology group is defined as follows: Write $\bar{Y}^n$ for $Y^n\cup B$, identify the cohomotopy group $\pi^n(\bar{Y}^n, \bar{Y}^{n-1})$ with the integral cochains $C^n(Y, B)$ and define δ: $C^n(Y, B)\rightarrow C^{n+1}(Y, B)$ as the coboundary homomorphism in the cohomotopy sequence of $(\bar{Y}^{n+1}, \bar{Y}^n, \bar{Y}^{n-1})$. For abelian G, the co-cochain groups

$$C_n(Y, B; G) = \mathrm{Hom}\,[C^n(Y, B); G]$$

lead in the usual way to the co-cohomology groups $H_n(Y, B; G)$. Next, let $k\geqq 2$, $\dim (X-A)\leqq 2k-2$, and $f:(X, A)\rightarrow(\bar{Y}^k, B)$. If j is the injection $(\bar{Y}^k, B)\rightarrow(\bar{Y}^k, \bar{Y}^{k-1})$, the map jf induces a homomorphism $z_k(f)$ of $\pi^k(\bar{Y}^k, \bar{Y}^{k-1})$ $(=C^k(Y, B))$ into $\pi^k(X, A)$. Thus, $z_k(f)\in C_k(Y, B; \pi^k(X, A))$, it is a co-cocycle, and is called the first obstruction to compressing f into $\bar{Y}^{k-1}$. It is then shown that $z_k(f)$ has properties analogous to those of the usual obstruction cocycle: (1) If $\bar{Y}^{k-1}$ is m-connected, $m\leqq k-1$, and

$$\dim (X-A)\leqq k+m-1,$$

then f is compressible into $\bar{Y}^k$ whenever $z_k(f)$ is the zero homomorphism; and (2) $f:(X, A)\rightarrow(\bar{Y}^k, B)$ is homotopic rel A in $\bar{Y}^{k+1}$ to a map into $\bar{Y}^{k-1}$ if and only if $z_k(f)$ is co-cobounding. This last result is shown to contain a theorem of Pontrjagin [Izv. Akad. Nauk SSSR. Ser. Mat. 13 (1949), 193–200; MR 11, 122, 380].

J. Dugundji.

Kiang, Tsai-han, and Sun, Yi-fone. Note on the homology groups of the fiber bundles over a sphere. J. Chinese Math. Soc. (N.S.) 1 (1951), 247–256. (Chinese summary)

In this paper, the authors prove several theorems on the homology groups of a fibre space whose base space is a sphere. These theorems are all consequences of either the exact sequence of Wang alone or of the exact sequence of Wang and the exact sequence of Gysin [for information on these two exact sequences, see J. Leray, J. Math. Pures Appl. (9) 29 (1950), 169–213; MR 12, 521; or J.-P. Serre, Ann. of Math. (2) 54 (1951), 425–505; MR 13, 574].

W. S. Massey (Providence, R.I.).

Gordon, William L. Locally-finitely-valued cohomology groups. Proc. Amer. Math. Soc. 6 (1955), 656–662.

It is known that for a compact Hausdorff space the Alexander-Kolmogoroff cohomology groups can be defined by the finite-valued functions [Keesee, same Proc. 1 (1950), 418–422; MR 12, 196]. In the present paper the author shows that for a fully normal Hausdorff space the functions which are locally finite-valued (i.e. for each point there is a neighborhood of the point in which the function takes on only a finite number of values) suffice to determine these cohomology groups. It is also shown that for a locally compact Hausdorff space the Alexander-Kolmogoroff cohomology groups with compact support can be defined by the finite-valued cochains with compact support.

E. H. Spanier (Chicago, Ill.).

Bott, R., and Samelson, H. The cohomology ring of G/T. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 490–493.

Given any compact connected Lie group G with maximal torus T, the authors construct a manifold B of the same dimension as G/T, which is a multiple fibering with 2-spheres as fibres and whose integral cohomology ring $H^*(B)$ is determined in terms of the diagram of G. Then they prove the existence of a map f of degree one of B onto G/T. From this result and the fact that the cohomology ring of G/T with rational coefficients is generated by $H^2(G/T)$, it is shown that the integral cohomology ring $H^*(G/T)$ is isomorphic to the smallest subring of $H^*(B)$ which is additively a direct summand and which contains $f^*H^2(G/T)$.

C. T. Yang (Princeton, N.J.).

See also: Denniston, p. 194.

GEOMETRY

Fadini, Angelo. Osservazioni su un teorema elementare di geometria. Giorn. Mat. Battaglini (5) 3(83) (1955), 55–60.

The author begins by stating and proving Theorem A: If on the sides BC, CA, AB, of a triangle ABC three points D, E, F, are marked arbitrarily, the circles AEF, BFD, CDE, have a point P in common [A. Miquel, J. Math. Pures Appl. 3 (1838), 485–487, p. 486]. The given figure Φ is transformed, by an inversion of center O, into the figure Φ', and this figure is transformed into the figure Φ'' by projecting Φ' from a point S, outside the plane of Φ', upon a plane not parallel to that plane. The original figure Φ is also projected upon a sphere Σ from a point O of that sphere, thus forming a figure Φ'''.

For the three figures Φ', Φ'', Φ''', the author states the forms A′, A″, A‴, which the Theorem A takes on. Thus, for instance, we read: Theorem A″: Given a linear system ∞^3 of conics with I_1 and I_2 as basic points, and in it the net of conics passing through the point O, let γ_a, γ_b, γ_c be three conics determined by the points:

$$I_1, I_2, O, B, C;\ I_1, I_2, O, C, A;\ I_1, I_2, O, A, B.$$

On these three conics the points D, E, F, are marked, respectively, and the three conics γ_A, γ_B, γ_C of the system are considered determined, respectively, by the points:

$$I_1, I_2, A, E, F;\ I_1, I_2, B, F, D;\ I_1, I_2, C, D, E.$$

The latter three conics pass through the same point P, regardless of the manner in which the points D, E, F, are chosen on the former three conics.

In a second theorem (Theorem B) the author considers an additional Miquel triangle $D_1E_1F_1$ of the same point P for the triangle ABC, and treats Theorem B in the same manner as Theorem A. *N. A. Court* (Norman, Okla.).

★**Voronkov, I. M. Some properties of the n-lines of a triangle.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 59–65. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Consider a triangle ABC and a line through a vertex; this line is called an n-line of the triangle if its point of intersection with the opposite side divides this side in the ratio of the nth powers of the adjacent sides. A triangle has three n-lines, one through each vertex; the n-lines are defined for any real n, in particular for $n=0, 1, 2$ they are the medians, bisectors of the angles, symmedians respectively. The three n-lines are concurrent; let O_n be their point of intersection. The author studies the locus of the points O_n. Denote by a, b, c the sides of the triangle, and assume that $c>b>a$. If there exists an integer k such that $b^k=ac^{k-1}$, then the locus of the points O_n is an algebraic curve of order k. If $k=2$ this curve is an ellipse, its properties are discussed in some detail. The author also obtains results concerning polygons: if a polygon with n vertices can be inscribed in a circle and if the center of gravity of its vertices is the center of the circumscribed circle then the sum of the squares of all sides and all diagonals equals n^2R^2 where R is the radius of the circumscribed circle. Several references are made to a book by S. I. Zetel′ [Modern geometry of the triangle, Učpedgiz, Moscow, 1940] which seems to contain some of the proofs not given in the paper. *E. Lukacs* (Washington, D.C.).

Sydler, J.-P. Le triangle comme opérateur géométrique. Elem. Math. 10 (1955), 100–105.

The author considers a rigid triangle, moving in the Euclidean plane in such a way that two vertices describe given rational curves of orders n and p. He proves that the third vertex describes a curve of order $2np$. (The cases $n=1$, $p=1$ or 2 are illustrated by elegant diagrams.) He deduces that there may be as many as $2npr$ congruent triangles whose vertices lie respectively on three given rational curves of orders n, p, r. In particular, if a rigid triangle moves so that two vertices both describe the same rational curve of order n, the third vertex describes a curve of order $2n^2$. (He accidentally omits the 2.) He remarks (with another fine diagram) that when the rational curve is a hyperbola, it meets the octavic locus in twelve proper points, which yield twelve congruent triangles all inscribed in the same hyperbola.

H. S. M. Coxeter (Toronto, Ont.).

Moisil, Gr. C. Un analogon pour les surfaces du théorème de concurrence des hauteurs dans un triangle. Gaz. Mat. Fiz. Ser. A. 7 (1955), 293–294. (Romanian. Russian and French summaries)

Haimovici, M. Remarques sur les six familles de courbes sur une surface étudiées dans la note précédente par l'acad. Gr. C. Moisil. Gaz. Mat. Fiz. Ser. A. 7 (1955), 295–297. (Romanian. Russian and French summaries)

Thébault, Victor. Sur la géométrie du tétraèdre. Mathesis 64 (1955), nos. 3–4–5, supplément, 1–16.

Deaux, R. Polarité dans un terne de points. Mathesis 64 (1955), 87–93.

Ram Mandan. Shortest distance. Bull. Calcutta Math. Soc. 46 (1954), 237–238.

Hazay, I. Zeitgemässe Formeln für Reduktionen von winkeltreuen Zylinderprojektionen. Acta Tech. Acad. Sci. Hungar. 11 (1955), 185–216. (Russian, English and French summaries)

The reductions for cartographic projections presented provide corrections to be applied to the distances and azimuth angles on the projection to obtain the corresponding true values. These are developed for the general class of conformal cylindrical projections in an approximate form suitable for neighboring points. Numerical examples are given. *N. A. Hall* (New York, N.Y.).

★**Coxeter, H. S. M. Reelle projektive Geometrie der Ebene.** Übersetzt von W. Burau. Verlag von R. Oldenbourg, München, 1955. 190 pp. DM 18.60. Have English

Translation of the author's "The real projective plane", 2d ed. [Cambridge, 1955; MR 16, 1143].

Gordevskiĭ, D. Z. Multidimensional analogues of the hyperboloid. Uspehi Mat. Nauk (N.S.) 10 (1955), no. 3(65), 129–133. (Russian)

In a projective space P of dimension $mk+m+k$ let $m+2$ linear subspaces $V_1, \cdots, V_{m+2}$ of dimension k be given such that any $m+1$ of these V_i span all of P. Through each point $p_i \in V_i$ there passes an m-dimensional linear subspace L_m of P which intersects all the other V_j. Such an L_m intersects each V_j in exactly one point p_j, and two different L_m intersecting all V_i do not intersect each other, so that the original L_m through p_i is uniquely determined. If p_i traverses a line in V_i, then each p_j traverses a line in V_j and the L_m traverses a $(2m+1)$-dimensional linear space.

H. Busemann (Los Angeles, Calif.).

Fadini, Angelo. Geometrie affini e geometrie metriche generalizzate. Ricerca, Napoli 5 (1954), no. 4, 18–26.

Extending his previous work on the plane [same Ricerca 5 (1954), no. 1–2, 57–64; MR 15, 818], the author makes some general remarks on linear spaces with unusual choices of the set of "points at infinity" and the associated groups of transformations with invariants. As an example, he considers in some detail the generalized 3-dimensional affine space S_3 in which the points at infinity consist of a pair of skew lines. There are numerous alternative possibilities as to parallels, perpendiculars, etc. For instance, there are three types of parallelism for two planes: they may have in common at infinity a line or two points or a

single point; correspondingly for lines, etc. In some cases distance and angle can be defined, in other cases not.

P. M. Whitman (Silver Spring, Md.).

Lombardo-Radice, Lucio. Sul rango dei piani grafici finiti a caratteristica 3. Boll. Un. Mat. Ital. (3) **10** (1955), 172–177.

In einer graphischen Ebene sei der Schnittpunktsatz S erfüllt, der in der Streckenrechnung von M. Hall besagt, dass $3=0$ ist, d.h. in einem cartesischen Koordinatensystem, dass die Punkte $(0, 2)$, $(1, 0)$ und der unendlich ferne Punkt der Geraden $x=y$ collinear sind. In jeder projektiven Ebene über einem Körper der Charaktristiek 3 ist S erfüllt. Eine graphische Ebene, in der S gilt, heises kurz eine graphische Ebene der Charakteristik 3. Verf. fragt nach der Gesamtheit der Typen solcher Ebenen. Falls die graphische Ebene endlich ist und den Rang t hat, so dass sie $n=t^2+t+1$ Punkte und ebensoviele Geraden besitzt, zeigt Verf., dass dann $(t^2+t+1)(t+1)(t-1)^2t^3$ durch 2^4, durch 3^3 und durch 13 teilbar sein muss. Diese Aussage folgt aus der Auszählung aller vorhandenen nicht entarteten Vierecke; sie verteilen sich auf Klassen so, dass die Vierecke derselben Klasse eine graphische Ebene vom Range 3 erzeugen, die zur ebenen Koordinatengeometrie über GF(3) isomorph ist. Die Frage bleibt offen, ob t stets eine Potenz von 3 sein muss und ob alle endlichen graphischen Ebenen der Charakteristik 3 Désargues'sch sind. Das analoge Problem für graphische Ebenen der Charakteristik p wird aufgeworfen. *R. Moufang.*

Roeser, Ernst. Reelle elliptisch-hyperbolische Zusammenhänge in der nichteuklidischen Geometrie. Rad. Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **302** (1955), 15–26. (Serbo-Croatian. German summary)

In hyperbolic 3-space with space constant 1 consider two lines M, N which are perpendicular to each other at z, and in the plane π of M and N a right triangle z, a, b with $a \in M$, $b \in N$, and angles λ at a, μ at b. Let R, A', B' be rays perpendicular to π at z, a, b and on the same side of π, and denote by A, B the parallels to A', B' through z. The sphere about z whose radius r is determined by $\sinh r=1$, intersects the rays R, A, B in points z', a', b'. If λ' and μ' are the angles of the spherical triangle z', a', b' at a' and b', then $\lambda+\mu'=\frac{1}{2}\pi$, $\lambda'+\mu=\frac{1}{2}\pi$, $\frac{1}{2}\pi-P(za)$ (where $P(za)$ is the parallel angle belonging to the hyperbolic distance za) plus the spherical distance $z'a'$ equals $\frac{1}{4}\pi$, similarlyfor zb and $z'b'$. An analogous result is obtained for quadrangles. *H. Busemann* (Los Angeles, Calif.).

Papy, Georges. Espaces vectoriels et espaces homogènes. Bull. Soc. Math. Belg. **1954** (1955), 106–114.

Elementary remarks on the well-known relations between vector spaces and affine spaces. *J. Dieudonné.*

Longo, Carmelo. Su un tipo particolare di complessi lineari di piani in S_{3r-1}. Boll. Un. Mat. Ital. (3) **9** (1954), 150–153.

A contravariant trivector a_{ijk} in E_{3r} is but for a scalar factor a linear S_2-complex in S_{3r-1} (S_p=p-dimensional projective space). It consists of r independent blades if a coordinate system can be found such that the only nonzero components are a_{123}, a_{456}, $\cdots$. For this case two necessary and sufficient conditions are given in a geometric form, using the r principal planes (piani cardini) in S_{3r-1} that are formed each from $r-1$ of the blades.

J. A. Schouten (Epe).

Graeub, W. Der Jordansche Kurvensatz in der affinen Geometrie. Ann. Acad. Sci. Fenn. Ser. A. I. no. **181** (1955), 13 pp.

Using only three incidence axioms: (1) Two points determine a unique line; (2) On each line lie at least two points; (3) There are three noncollinear points, and four axioms of betweenness: (1) $S(ACB)\Rightarrow S(BCA)$; (2) For any three distinct collinear points exactly one of the following holds: $S(ABC)$, $S(BCA)$, $S(CAB)$; (3) For any two A, B there are C, D, E on the line AB so that $S(DAB)$, $S(ACB)$, $S(ABE)$; (4) The Axiom of Pasch, which asserts that a line meeting the interior of one side of a triangle meets one of the other two sides, the author derives the Jordan Separation Theorem for polygons. *S. Stein.*

★ **Borsuk, Karol, i Szmielew, Wanda. Podstawy geometrii.** **[The foundations of geometry.]** Państwowe Wydawnictwo Naukowe, Warsaw, 1955. 363 pp. zł. 25.00.

This book deviates from other works with the same title both in content and in the extent to which details often omitted in other books are carried out.

It begins with an introduction on the history of the subject and on the very elements of point-set topology. The first four chapters are concerned with absolute geometry, in particular chapter I (pp. 23–74) with the axioms of incidence and order. As an example of a topic usually omitted we mention the topology of the plane obtained by defining the interiors of triangles as neighborhoods. A theorem like the following is at this stage not quite trivial: If a_1, a_2, a_3, a_4 are the vertices of a convex quadrangle and p is the intersection of its diagonals, then a_i has a neighborhood U_i such that for $a_i' \in U_i$ the quadrangle a_1' a_2' a_3' a_4' is convex and has its intersection of the diagonals in a preassigned neighborhood of p. Chapter II (pp. 74–112) treats the axioms of congruence. In Chapter III (pp. 113–158) we find the continuity axioms, Legendre's theorems on the angle sums of triangles, and a proof that the space is now metric and homeomorphic to E^2 or E^3. Chapter IV (pp. 159–200) discusses models of absolute geometry and its noncategoricalness. Chapter V (pp. 201–213) outlines Euclidean geometry, whereas Chapter VI (pp. 214–273) treats hyperbolic geometry in great detail including the derivation of many of the more complicated formulae. Chapter VII (pp. 274–314) gives the foundations of plane and spatial projective geometry without continuity. The final Chapter VIII (pp. 315–346) deals with the continuity axioms of projective geometry and ends with a curiously detailed analysis of Hilbert's example for a non-Desarguesian geometry. A detailed index follows.

An English edition of this book might well be an answer to the problem of how to acquaint young students with exact reasoning, presenting at the same time a coherent course. *H. Busemann* (Los Angeles, Calif.).

Convex Domains, Extremal Problems, Integral Geometry

Rabin, Michael. A note on Helly's theorem. Pacific J. Math. **5** (1955), 363–366.

The much-proved theorem of Helly [Jber. Deutsch. Math. Verein. **32** (1923), 175–176] asserts that if F is a finite family of convex sets in E^n, having empty intersection, then for some $k\leq n$ there are $k+1$ members of F

whose intersection is empty. The present author first proves the theorem by induction on n for the case in which each member of F is a closed half-space, proceeds from there to the case of convex polytopes, and thence to the general case. His proof (like several others) applies to an n-dimensional affine space over an arbitrary subfield of the reals. The paper is concluded with a short deduction from Helly's theorem of Carathéodory's theorem on convex hulls. The dual relationship of these theorems has been recently studied also by Sandgren [Math. Scand. 2 (1954), 19–28; MR 16, 395]. *V. L. Klee, Jr.*

Rubinšteĭn, G. Š. On a method of investigation of convex sets. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 451–454. (Russian)

The points of a convex subset C of a real linear space are ordered as follows: $y \geqq x$ provided $x+t(y-x) \in C$ for some $t>1$; $y>x$ provided $y \geqq x$ and $\sim(x \geqq y)$. The ordering $\geqq$ is observed to be reflexive and transitive, and various other properties are described, some of which are employed to yield a "parametric representation" of linearly closed convex sets which do not contain an infinite strictly decreasing sequence of points. Also included are remarks on systems of linear inequalities, on positive linear functionals, and on an approximation problem of Haar. *V. L. Klee, Jr.* (Los Angeles, Calif.).

Lee, Tzer-Yee, Lin, Jen-Sheng, Tong, Kwang-Chong, and Zhang, Ming-Yng. A solution of Bang's "plank problem". J. Chinese Math. Soc. **2** (1953), 139–143. (Chinese. English summary)

Notation: L is a convex body in R_n. A strip in R_n is the space bounded by two parallel hyperplanes Π^+ and Π^-; its complementary set consists of two half-spaces H^+ and H^- bounded by Π^+ and Π^-, respectively. The vector $2\bar{u}$, of length h, from Π^- to Π^+ is perpendicular to both hyperplanes, and the vector $\bar{a}$ from the coordinate origin O to the mid-hyperplane of the strip is parallel to $2\bar{u}$; further $c=-\bar{a}\cdot u$ denotes the scalar product of $\bar{a}$ and $\bar{u}$. Hence if r denotes the vector from O to any point on Π^+, then $\bar{r}\cdot\bar{u}+c=\bar{u}\cdot\bar{u}$; for this reason the pair $(\bar{u}, c)$ may serve to fix the strip. The width k of L in the direction of $2\bar{u}$ is the maximum distance of any two points A, B of L for which $\overrightarrow{AB}$ is parallel to $2\bar{u}$. Let now $(\bar{u}_i, c_i)$, $i=1, 2, \cdots, h$, be finitely many strips bounded by the hyperplanes Π_i^+ and Π_i^- and of widths $h_i=2|\bar{u}_i|$, and let H_i^+ and H_i^- be the complementary closed half-spaces; assume that the union of the strips covers L. A conjecture by T. Bang [Proc. Amer. Math. Soc. 2 (1951), 990–993; MR 13, 769] states that (1) $\sum_{i=1}^k h_i/k_i \geqq 1$. The authors give a proof based on the following two lemmas. (I) If k and k' are the widths of L in the (equal or different) directions of $2\bar{u}$ and $2\bar{u}'$, then the width of $(L-\bar{u})\cap(L+\bar{u})$ in the direction of $2\bar{u}'$ is not less than $k'(1-h/k)$. (II) Let $e_1, \cdots, e_p$ be signs $+$ or $-$, and let $P_{e_1\ldots e_p}=\bigcap_{i=1,2,\ldots,p} H_i^{e_i}$. Then

$$\bigcup_{e_1,\ldots,e_p}(P_{e_1\ldots e_p}-e_1\bar{u}_1-\cdots-e_p\bar{u}_p)=R_n.$$

Let now Bang's conjecture (1) be false. Since the p strips cover L, it follows easily from (II) that

$$K=\bigcap_{e_1,\ldots,e_p}(L-e_1\bar{u}_1-\cdots-e_p\bar{u}_p)=0.$$

On the other hand, a repeated application of (I) shows step by step that the width of $\bigcap_{e_1,\ldots,e_i}(L-e_1\bar{u}_1-\cdots-e_i\bar{u}_i)$ in the direction of $2u_j$ is not less than

$$k_j(1-h_1/k_1-\cdots-h_i/k_i)>0,$$

whence $K\neq 0$ for $i=p$. *K. Mahler* (Zbl. 52, 394).

Eggleston, H. G. Sets of constant width contained in a set of given minimal width. Mathematika **2** (1955), 48–55.

It is shown that every plane convex set of width λ contains a convex subset of constant width $\geqq\lambda/(3-\sqrt{3})$, this bound being sharp for an equilateral triangle. *W. Gustin* (Bloomington, Ind.).

Shimrat, Moshe. Simple proof of a theorem of P. Kirchberger. Pacific J. Math. **5** (1955), 361–362.

A new proof of the theorem: two finite point sets in n-space can be separated by a hyperplane if and only if every $k+1$ points of one can be so separated from every $j+1$ of the other, for all j, k such that $k+j\leqq n$. *D. Gale* (Providence, R.I.).

Grünbaum, B. On a theorem of L. A. Santaló. Pacific J. Math. **5** (1955), 351–359.

The author proves the following theorem: Let $S_1, \cdots, S_n$ be a family of connected sets in the plane such that for any $i<n$, $\bigcup_{j\leq i} S_j$ and $\bigcup_{j>i} S_j$ can be strictly separated by a line. Then either (1) there is a line which intersects all the S_i, or (2) there are three sets in the family which cannot be intersected simultaneously by any line. In the case where the separating lines in the hypothesis above are parallel, the author gets the stronger result: Given a finite family of connected sets in the plane any two of which can be separated by a line of fixed direction, then again (1) or (2) above. Both the above theorems can be proved also for the case of infinite families of compact sets in a standard manner. [Note: Klee [Proc. Amer. Math. Soc. 5 (1954), 639–641; MR 16, 162] has also given a proof of the second theorem mentioned above.] *D. Gale.*

Hsieh, L. T. On doubly piercing lines of space pentagons. J. Chinese Math. Soc. (N.S.) **1** (1951), 395–408. (Chinese summary)

A straight line in E^3 is said to pierce the pentagon P with vertices a_i $(i=1, \cdots, 5)$, doubly if it is encircled twice by P. Both analytic conditions, in terms of the calculus of extension, and geometric ones are given. The following are samples for the latter. P possesses doubly piercing lines if and only if no segment $a_{i-1}a_{i+2}$ has a common point with the 2-simplex $a_ia_{i+2}a_{i+3}$. There is a doubly piercing line through a given point q if and only if each of the pentagons $qa_ia_{i+1}a_{i+2}a_{i+3}$ possesses doubly piercing lines passing through a given neighborhood of q. *H. Busemann* (Los Angeles, Calif.).

Fenchel, W. On the convex hull of point sets with symmetry properties. Rend. Mat. e Appl. (5) **14** (1955), 355–358.

Let Γ be a group of orthogonal transformations of euclidean n-space E^n. The set of all points fixed under Γ is then a linear subspace, say F of dimension f, in E^n. Consider a set M in E^n setwise fixed under Γ. It is shown that any point on F in the convex hull of M lies in the convex hull of the union of the orbits under Γ of $f+1$ or fewer points from M. For subsets M of F this result collapses to the well known classical one, since the orbit of a point from F collapses to just that point itself. *W. Gustin* (Bloomington, Ind.).

Masotti Biggiogero, Giuseppina. Su alcune formule di geometria integrale. Rend. Mat. e Appl. (5) **14** (1955), 280–288.

Let R be a secant which cuts an oval at points A_1 and

A_2; let P be the point of intersection of the tangents to the oval at A_1 and A_2; let D be the diameter of the circle circumscribed about the triangle A_1A_2P, and let ϱ_1 and ϱ_2 respectively be the radii of curvature of the oval at A_1 and A_2. Let dR be the density of secants R and $dP=dxdy$. Then it is shown that

$$dR=\frac{\varrho_1\varrho_2}{D^3}dP.$$

This formula is used to recast various results of Crofton, Santaló, and Hurwitz regarding ovals. It also shows a relationship between these formulas and the formula of Lebesgue. *C. B. Allendoerfer* (Seattle, Wash.).

Ueno, Seitarô. On the densities in a two-dimensional generalized space. Mem. Fac. Sci. Kyûsyû Univ. Ser. A. **9** (1955), 65–77.

This paper is a contribution to the integral geometry of a two-dimensional Finsler space. The methods used are modeled on the work of Haimovici [Ann. Sci. Univ. Jassy. Sect. I. **23** (1937), 57–74] and Santaló [Duke Math. **16** (1949), 361–375; MR **10**, 732]. The density dG of a vector field $a(x)$ is first defined, where $a(x)$ is assumed to be of unit length with respect to itself. This density is zero if $a(x)$ is a unit tangent vector field of geodesics. It is also the density of the geodesic G with directive element $(x, a(x))$.

This leads to Crofton's formula concerning the number of intersections of G with a rectifiable curve of length L. Then the kinematic density for curves, $dK=2[dGds]$ is defined, and this does not depend on the vector field $a(x)$ used to define dG. This leads to Poincaré's formula for the number of intersections of a fixed curve of length L_0 and a congruent curve of length L. *C. B. Allendoerfer*

See also: Rubinšteĭn, p. 133.

Differential Geometry

★**Santaló, L. A. Questions of the affine differential geometry of surfaces.** Segundo symposium sobre algunos problemas matmáticos que se están estudiando en Latino América, Julio, 1954, pp. 21–33. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

This article is an exposition of the theory of the affine differential geometry of surfaces. A full treatment of the subject is contained in volume II of Blaschke's Vorlesungen über Differentialgeometrie [Springer, Berlin, 1923]. Santaló presents the essentials of this theory by means of the notation and methods of E. Cartan.

C. B. Allendoerfer (Seattle, Wash.).

Şemin, F. Sur une propriété caractéristique des surfaces à courbure moyenne constante. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19** (1954), 140–147.

At each point of a G-curve on a real surface the normal section in the direction of the G-curve is superosculated by its circle of curvature. The author shows that a G-curve can also be defined by the condition $L=0$, where $L=d\varrho_n/ds-2\varrho_g\tau_g$ is the Laguerre function of direction. Here s, ϱ_n, ϱ_g, τ_g are respectively the arc length, normal curvature, geodesic curvature, and geodesic torsion. The spheres, cylinders of revolution, and planes are the surfaces on which all curves are G-curves. In general there are three families of G-curves on a surface S and the necessary and sufficient condition that these families intersect at 120° is that S have constant mean curvature. On such a surface the D-curves are orthogonal trajectories of the G-curves and the same theorem is true for them. [On the D-curves, the Darboux function of direction vanishes, $D=d\tau_g/ds+(2\varrho_n-W)\varrho_g=0$. Here $W/2$ is the mean curvature.]

In the important equation (3—1), replace the second "+" symbol by "cos α". *A. Schwartz.*

Mihăileanu, N. N. Une extension de la relation Rodrigues. Gaz. Mat. Fiz. Ser. A. **7** (1955), 313–314. (Romanian. Russian and French summaries)

Sakellariou, Neilos. Relation between the principal radii of curvature and the unit normal of a surface. Prakt. Akad. Athēnōn **28** (1953), 392–395 (1954). (Greek)

If a surface S in E^3 is referred to the lines of curvature as coordinate lines, then the formulae of Olinde Rodrigues give a relation between the unit normal of S and its principal radii of curvature $R_1(u, v), R_2(u, v)$. This paper derives other relations of which the following is typical: If e, f, g are the coefficients of the first fundamental form of the focal surface (referred to the same coordinates u, v) belonging to the congruence of normals of S, then

$$e\frac{\partial R_1}{\partial u_2}:\frac{\partial e}{\partial u_2}+g\frac{\partial R_2}{\partial u_1}:\frac{\partial g}{\partial u_1}=0.$$

H. Busemann (Los Angeles, Calif.).

Sakellariou, Neilos. Spherical images of different types of a given surface. Prakt. Akad. Athēnōn **29** (1954), 380–382 (1955). (Greek)

The moving frame F of Frenet belonging to a curve C in E^3 is replaced by a frame F_φ obtained from F by revolving F about the tangent of C through a fixed angle φ. Unit-vectors from the origin parallel to the axes of F generate two new spherical curves. Their relations to C are investigated. *H. Busemann.*

Čech, E. Remarques au sujet de la géométrie différentielle projective. Acta Math. Acad. Sci. Hungar. **5** (1954), supplementum, 137–144. (Russian summary)

This is a translation of a paper that previously appeared in Hungarian [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3** (1953), 219–225 MR **15**, 467]. Besides general remarks on the developments in geometry from Bolyai and Lobachevsky to the present, the paper gives a brief account of the problem of projective deformation of surfaces which was initiated by Fubini [Rend. Circ. Mat. Palermo **41** (1916), 135–162] and which was solved in its most essential features by Cartan [Ann. Sci. Ecole Norm. Sup. (3) **37** (1920), 259–356]. Čech attacked the problem again in 1952 and pretty well exhausted it in a series of eight papers [Čehoslovack. Mat. Ž. **2(77)** (1952), 91–107, 109–123, 125–148, 149–166, 167–188, 297–331; **3(78)** (1953) 123–137; **4(79)** (1954), 143–174; MR **16**, 71, 748].

M. S. Knebelman (Pullman, Wash.).

Ramos e Costa. Note on the projective differential geometry of surfaces. Univ. Lisboa. Revista Fac. Ci. A. (2) **4** (1955), 191–194. (Portuguese)

Mediante semplici considerazioni l'A. ritrova le forme differenziali di Fubini e le equazioni di Wilczynski relative ad una superficie non rigata. *C. Longo* (Roma).

Korovin, V. I. The system R in four-dimensional projective space. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 797–799. (Russian)

In order to study hypersurfaces with triply conjugate systems of straight lines in a 4-dimensional projective space, the author considers a system of partial differential equations of Laplacian type: $X_{uv}=a_{12}{}^1X_u+a_{12}{}^2X_v+a_{12}X$ and two similar ones for X_{uw} and X_{vw}, u, v, w being the independent variables. It is easily verifiable that $f_{12}=a_{12}{}^1a_{12}{}^2+a_{12}-a_{12v}{}^2$ and $f_1=a_{12}{}^1-a_{23}{}^3$, and seven more similar expressions, are invariant under the transformation $X=\varrho X'$. Moreover, the conditions of integrability of the original system are expressible in terms of the f's. The quantities $X_{12}=X_u-a_{12}{}^2X$, and five others, satisfy equations of Laplacian type and their invariants are expressible in terms of the f's. The above transformation is called a Laplace transformation. Considering the hypersurface in tangential coordinates one obtains similar invariants F_{12}, F_1, etc. The first theorem states that if $f_{12}=F_{21}=-f_1F_2$ and ten more such relations hold for a given hypersurface, they hold for all Laplace transformations of the surface. Another theorem states that hypersurfaces satisfying the above 12 relations exist and that the solution of the differential equations depends on 15 arbitrary constants. *M. S. Knebelman.*

Korovin, V. I. Closed Laplace sequences. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 605–606. (Russian)

The paper is concerned with a triply conjugate system of surfaces in a projective space of three dimensions described by A_0 which is a vertex of a moving tetrahedron with vertices A_i ($i=0, 1, 2, 3$). An infinitesimal motion is described by $dA_i=\omega_i{}^kA_k$, the lines of intersection being $\omega_0{}^2=\omega_0{}^3=0, \omega_0{}^1=\omega_0{}^3=0, \omega_0{}^1=\omega_0{}^2=0$. If A_{ij} ($i,j=1,2,3$), $i\neq j$, denotes the Laplace transform of A_0 in direction of line i, transposed along line j, then the points A_{ij} also describe a triply conjugate system. A sequence $\cdots, A_{2121}, A_{21}, A_0, A_{12}, A_{1212}, \cdots$ is called a principal Laplace sequence with similar constructions for 2, 3 and 1, 3. A principal sequence is closed if it consists of 4 line segments with vertices A_{21}, A_0, A_{12}, A_{1212} ($A_{1212}\equiv A_{2121}$). A derived sequence is constructed by using some of the derived points. If the three principal sequences are closed so are all the derived sequences. The main result is that in this case there exist 16 points each of which describes a triply orthogonal system of surfaces and each of these points may be obtained from any of the others by at most two Laplace transformations.

M. S. Knebelman (Pullman, Wash.).

Marcus, F. Sur un système de lignes isothermiquement conjugées sur une surface. Com. Acad. R. P. Romåne **3** (1953), 175–178. (Romanian. Russian and French summaries)

Let D, $D'=0$, D'' be the coefficients of the second fundamental form of a surface referred to a system of conjugate lines as parameters. Let h and k be the invariants of the Gauss formula

$$x_{uv}=\left\{ {1\ 2 \atop 1} \right\}x_u+\left\{ {1\ 2 \atop 2} \right\}x_v$$

satisfied by the generic point of the surface. Let $\bar{h}$ and $\bar{k}$ be the invariants of the equation

$$N_{uv}=\left\{ {1\ 2 \atop 1} \right\}'N_u+\left\{ {1\ 2 \atop 2} \right\}'N_v-fN$$

satisfied by the normal to the surface,

$$\left\{ {i\ j \atop e} \right\}'$$

being Christoffel symbols referred to the third fundamental form $e\,du^2+2f\,du dv+g\,dv^2$. Then the author establishes the formula $\partial^2 \log (D/D'')/\partial u\partial v=\bar{k}-\bar{h}+k-h$, and concludes that $D=\pm D''$ if $\bar{k}-\bar{h}+k-h=0$, and conversely. *A. Schwartz* (New York, N.Y.).

Marcus, F. Sur les surfaces isothermes-asymptotiques. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6** (1954), 819–821. (Romanian. Russian and French summaries)

A theorem of T. Mihăilescu [Acad. Repub. Pop. Romåne. Stud. Cerc. Mat. **1** (1950), 374–392, p. 376; MR **16**, 514] is generalized as follows. In a three-dimensional projective space a surface S is isothermally asymptotic if the congruence generated by a canonical line of the first kind of S is stratifiable in one direction, the stratifying congruence being generated by the corresponding canonical line of the second kind of S. *C. C. Hsiung.*

Marcus, F. Sur une classe de surfaces. Com. Acad. R. P. Romåne **2** (1952), 397–400. (Romanian. Russian and French summaries)

E. Cartan showed that for an E-surface the asymptotic lines of the two Laplace transforms correspond, but that this is not a characteristic property of E-surfaces [Bull. Sci. Math. (2) **68** (1944), 41–50; MR **7**, 78]. Here it is shown that the condition that the asymptotic lines shall correspond is $hk=\bar{h}\bar{k}$, $h, k, \bar{h}, \bar{k}$ being the invariants of the point and tangential equations of Laplace. For the E-surfaces $h=\bar{h}$, $k=\bar{k}$. For R-surfaces $h=\bar{k}$, $k=\bar{h}$. The surfaces for which the tangents to the lines of a conjugate network describe Waelsch line congruences also belong to this class of surfaces: $h=-\bar{k}$, $k=-\bar{h}$.

A. Schwartz (New York, N.Y.).

Marcus, F. Sur les surfaces de coïncidence. Com. Acad. R. P. Romîne **5** (1955), 307–310. (Romanian. Russian and French summaries)

Fubini and Čech have pointed out that if a surface for which the canonical line pencils coincide is referred to its asymptotic lines in a suitable way, then the normal coordinates of a generic point are solutions of the system of equations $x_{uu}=x_v+p_{11}x$, $x_{vv}=x_u+p_{22}x$ [Geometria proiettiva differenziale, t. I, Zanichelli, Bologna, 1926, p. 157]. Here it is shown that such a surface is minimal projective if and only if p_{11} and p_{22} are constant.

A. Schwartz (New York, N.Y.).

Myller, A. La transformation de Legendre vue géométriquement. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Ser. I. **5** (1954), no. 3–4, 1–12. (Romanian. Russian and French summaries)

The author gives a simple geometric construction for the Legendre transformation formulas used in ordinary differential equations; from a curve $\eta=\eta(\xi)$ we are led to $y=y(x)$ by using $x=d\eta/d\xi$, $y=\xi d\eta/d\xi-\eta$. A simple method for finding invariants of this transformation is given: let F be a symmetric function of two variables. Then $F(\xi,\eta')=F(\eta',\xi)=F(x,y')$. For instance, $\xi\eta'=\eta'\xi=y'x$, and thus the subtangent with respect to the $O\eta$-axis is left invariant. Next a method for finding curves which are left invariant is given. By studying the Legendre transforms of a one-parameter family of curves, the

author finds two families of curves with the property that the curves of the first family have parallel tangents at the points where they meet an arbitrary curve of the second family. If the family with which we start is a pencil of curves, then the second of the two families we get will consist of straight lines and any three curves of the first family will cut the lines of the second family in proportional segments. The problem of finding curves whose Legendre transforms will preserve specific properties not preserved by all Legendre transforms is also considered. *A. Schwartz* (New York, N.Y.).

Cantoni, Lionello. Sulle trasformazioni puntuali fra spazi proiettivi sovrapposti nell'intorno di un punto unito. Boll. Un. Mat. Ital. (3) **10** (1955), 212–223.

Una trasformazione puntuale T tra due spazi proiettivi sovrapposti S_r determina in un punto unito O una omografia ω non degenere tra le stelle, sovrapposte, di centro O.

Se l'invariante relativo ad una direzione unita nella ω è diverso da uno, sulla direzione è determinata intrinsecamente una proiettività non parabolica. Per mezzo delle dette proiettività, della configurazione caratteristica e delle omografie tangenti alla T in O, l'Autore determina, per $r=3$, nei vari casi che può presentare l'omografia ω in relazione alle direzioni unite, riferimenti intrinseci per la T ed il significato degli invarianti relativi al 1° e 2° ordine. Il riferimento intrinseco dipende dall'intorno del 2° ordine di O, a meno che esista un altro punto unito infinitamente vicino ad O nel qual caso occorre l'intorno del 3° ordine. Indicata con A e $\bar{A}$ una coppia di punti corrispondenti nella T, l'A. determina l'ipersuperficie L luogo di punti A per i quali esiste un iperpiano unito nell'omografia ω che T induce fra le stelle di centro A e $\bar{A}$, e caratterizza le singolarità che la L presenta nel punto O nei vari casi. *C. Longo* (Roma).

Franceschi, Odoardo. I tipi di superficie rigate razionali del 5° ordine dell'ordinario spazio, di H. A. Schwarz, dedotti come proiezioni della superficie rigata razionale normale del 5° ordine di S_6. Giorn. Mat. Battaglini (5) 3(83) (1955), 107–113.

Barner, Martin. Über gewisse Kurventripel auf Regelflächen. Arch. Math. **6** (1955), 223–229.

Continuing the work of L. Dalmasso [Boll. Un. Mat. Ital. (3) **9** (1954), 66–73; MR **15**, 826], the author studies associated triples of curves on a ruled surface S in a three-dimensional projective space by means of plane hyperbolic geometry. It is shown that under a certain mapping of S onto a hyperbolic plane the invariant parameter, introduced by the author [Math. Z. **56** (1952), 409–442; MR **14**, 687], for a pair of curves on S corresponds to the hyperbolic arc length of the image curve of the given pair, and also that a pair of curves on S is associated if its hyperbolic image is a circle with hyperbolic curvature $\frac{1}{2}$. *C. C. Hsiung.*

Vincensini, Paul. Sur quelques nouveaux aspects géométriques des transformations conformes du plan. Bull. Sci. Math. (2) **79** (1955), 42–48.

Verfasser berichtet über die Weiterentwicklung der von ihm angeregten Untersuchungen über (H)-Kongruenzen [Bull. Sci. Math. (2) **68** (1944), 60–72; MR **7**, 78] durch Simonart und Alardin [Acad. Roy. Belg. Bull. Cl. Sci. (5) **35** (1949), 602–613; MR **11**, 396] sowie durch S. Süray [Comm. Fac. Sci. Univ. Ankara. Sér. A. **4** (1952), 40–53; MR **15**, 157]. Eine (H)-Kongruenz entsteht, indem man die Normalen $\overrightarrow{MN}$ einer harmonischen Fläche S, parallel mit sich in die Punkte $m(x, y)$ der Grundebene von S, d.h. in die Punkte ihres Grundrisses überträgt. Diese Kongruenzen sind spezielle Ribaucourkongruenzen, die Grundrißebene (Grundebene) ist die Mittelebene. Wählt man dann im Raum einen festen Punkt und legt durch diesen wiederum alle zu den Normalen $\overrightarrow{MN}$ parallelen Geraden, so erhält man neben $m(x, y)$ neue Durchstoßpunkte $\mu(x, y)$ mit der Grundebene. Durch die Zuordnung $m\to\mu$ ensteht die allgemeinste mittelbare (inverse) konforme Abbildung der Ebene. Die weitere Untersuchung fügt nunmehr der harmonischen Fläche S ihre harmonisch konjugierte S' hinzu und betrachtet die einparametrige Schar der assoziierten Flächen

$$z_\alpha = z \cos\alpha + z' \sin\alpha.$$

Durch Drehung der Strahlen des Systems (H) um den gleichen Winkel α um die Senkrechten in den Durchstoßpunkten der Grundebene ergibt sich dann aus (H) ein System (H_α) der gleichen Familie ((H) geht durch die Transformation T_α in (H_α) über). Simonart und Alardin haben nun neben der Kongruenz (H) solche (P) betrachtet, die man erhält, indem man die gemeinsamen Senkrechten zweier konjugiert harmonischer Flächen in zwei beliebigen korrespondierenden Punkten (gleicher Projektion $m(x, y)$ auf die Grundebene) auswählt und durch jeden Punkt m die, zu diesen gemeinsamen Senkrechten, gehörige Parallele zieht. Süray hat das System (P) noch insofern vervollständigt, indem er diese Parallelen aus irgend zwei harmonisch assoziierten Flächen gewinnt. Alle diese Kongruenzen (P) sind wieder Ribaucoursche Kongruenzen. Verfasser zeigt, wie man weitere Ergebnisse von Simonart, Alardin und Süray durch die Methode der T_α-transformationen gewinnen kann. Zur Korrespondenz $m\to\mu$ gesellt sich jetzt die Korrespondenz $\mu\to\mu_\alpha$ und zu den mittelbaren konformen Abbildungen kommen unmittelbare (directe). Schließlich werden noch die Transformationen studiert, welche (H)-Kongruenzen in (P)-Kongruenzen verwandeln. *M. Pinl* (Köln).

Frey, Annemarie, und Strubecker, Karl. Die Transformationstheorie der quadratischen Linienkomplexe [(11) (22)]. II. J. Reine Angew. Math. **194** (1955), 1–20.

The authors here continue the study begun in part I [same J. **193** (1954), 209–238; MR **16**, 745] of the transformation theory of quadratic line complexes of type [(1 1) (2 2)]. A one-parameter group A_κ of non-projective automorphisms is obtained. It is shown that the transformations A_κ effect a permutation of the potential surfaces Π_m which are defined by the equations $z=m\mathfrak{J}[(x+iy)^{1/m}]$. The effects of polarization with respect to the quadric $\Pi_{\frac{1}{2}}$ on these surfaces and on curves of the complex are studied. Constructions are given for the asymptotic lines on the potential surfaces in terms of curves of the complex. A class of surfaces related to the complex of which the potential surfaces are a special case is defined and studied. *S. B. Jackson* (College Park, Md.).

Blanuša, Danilo. Über die Einbettung hyperbolischer Räume in euklidische Räume. Monatsh. Math. **59** (1955), 217–229.

A C_∞ (but not analytic) imbedding of the hyperbolic plane H^2 in the six-dimensional euclidean space E^6, and a C_∞ (but not analytic) imbedding of H^n in E^{6n-5} are given explicitly. *H. Busemann.*

Heinz, Erhard. Über Flächen mit eineindeutiger Projektion auf eine Ebene, deren Krümmungen durch Ungleichungen eingeschränkt sind. Math. Ann. **129** (1955), 451–454.

Simple proofs, with some improvement on the constants are given for the following results of S. Bernstein [Math. Ann. **69** (1910), 82–136; Ann. Sci. Ecole Norm. Sup. (3) **27** (1910), 233–256] and Efimov [Dokl. Akad. Nauk SSSR (N.S.) **93** (1953), 393–395, 609–611; MR **15**, 740]: Let $z(x, y)$ be defined in $x^2+y^2<R^2$ and of class C^2. If $H(x, y)$ and $K(x, y)$ are the mean and Gauss curvatures of the surface $z=z(x, y)$, then $|H|\geqq\alpha>0$ implies $R\leqq\alpha^{-1}$. If $K\geqq\alpha>0$, then $R\leqq\alpha^{-\frac{1}{2}}$. If $K\leqq-\alpha<0$, then $R\leqq e(3/\alpha)^{\frac{1}{2}}$.

H. Busemann (Los Angeles, Calif.).

Riemannian Geometry, Connections

Prvanovitch, Mileva. A field of vectors along a curve of sub-space of a Riemannian space. Srpska Akad. Nauka. Zb. Rad. **43.** Mat. Inst. **4** (1955), 135–143. (Serbo-Croatian. English summary)

Soit $V_n \subset V_m$ où V_m est l'espace de Riemann avec les points y^α $(\alpha=1, \cdots, m)$. Les coordonnées des points de V_n soient x^i $(i=1, \cdots, n)$. L'auteur considère le cercle géodésique de V_n donné par l'équation

$$\frac{D^3x^i}{ds^3}+g_{jk}\frac{D^2x^j}{ds^2}\frac{D^2x^k}{ds^2}\frac{Dx^i}{ds}=0$$

(D est le symbol de la dérivée absolue, g_{jk} est le tenseur fondamental de V_m).

Le vecteur

$$v^i=\frac{D^3x^i}{ds^3}+g_{jk}\frac{D^2x^j}{ds^2}\frac{D^2x^k}{ds^2}\frac{Dx^i}{ds}$$

est égal à zéro le long du cercle géodésique de V_n.

Le vecteur

$$\mu^\alpha=\frac{D^3y^\alpha}{ds^3}+a_{\beta\gamma}\frac{D^2y^\beta}{ds^2}\frac{D^2y^\gamma}{ds^2}\frac{Dx^\alpha}{ds}$$

de V_m correspond à v^i du cercle géodésique de l'éspace $V_n \subset V_m$ ($a_{\beta\gamma}$ est le tenseur fondamental de V_m). L'auteur établie la relation entre le vecteur μ^α et v^i et montre quelques consequences de la relation trouvée. P.e.: Le cercle géodésique de l'espace $V_m \subset V_n$, qui est totalement géodésique, est aussi le cercle géodésique de l'espace ambiant V_n.

F. Vyčichlo (Prague).

Sun, Peng-Wang. On the geometry of symplectic transformations. I. Symmetric Riemannian spaces of symplectic anti-involutions. J. Chinese Math. Soc. (N.S.) **1** (1951), 296–331. (Chinese summary)

The paper verifies by explicit calculations various properties pertaining to the metric $ds^2=\sigma(d\overline{T}dT)$ in the space E of unitary symplectic matrices T with $T\overline{T}=I$, and also properties pertaining to the (somewhat vaguely defined) metric $ds^2=|\sigma(d\overline{T}dT)|$ in the space R_1 of all symplectic matrices T with $T\overline{T}=I$.

The space E is simply connected and "elliptic" under the transformations $T^*=\overline{P}TP^{-1}$ (P is any unitary symplectic matrix) which also reduce all geodesics into the form

$$T(t)=[e^{i\lambda_1 t}, \cdots, e^{i\lambda_n t}, e^{-i\lambda_1 t}, \cdots, e^{-i\lambda_n t}].$$

In the space R_1 however, under the corresponding group of transformations, the geodesics fall into five separate classes two of which are

$$T(t)=\begin{pmatrix} H(t) & 0 \\ 0 & \overline{H}(t) \end{pmatrix},\quad T(t)=\begin{pmatrix} 0 & iH(t) \\ i\overline{H(t)} & 0 \end{pmatrix},$$

where $H(t)=[e^{i\lambda_1 t}, \cdots, e^{i\lambda_n t}]$.

S. Bochner.

Stojanovitch, Rastko. A note on a theorem of E. Cartan on the groups of stability. Tensor (N.S.) **5** (1955), 54–55.

The following theorem is proved: When a Riemannian space V_n admits the group G_r of motions which is at the same time the group of stability of a certain point 0 in V_n, then each of the subgroups G_1 of G_r is the group of stability of an $(n-2)$-dimensional totally geodesic subspace of V_n.

C. B. Allendoerfer (Seattle, Wash.).

Stojanovitch, Rastko. An inversion of a theorem of L. P. Eisenhart on the hypersurfaces in Riemannian spaces. Tensor (N.S.) **5** (1955), 56–57.

The following partial converse of a theorem of Eisenhart is proved: If a Riemannian space admits a family of geodesically parallel hypersurfaces and if there is a conformal correspondance between them, then the lines of curvature of the hypersurfaces are indeterminate. Moreover, the hypersurfaces are of constant curvature if the Riemannian space has constant curvature.

C. B. Allendoerfer (Seattle, Wash.).

Prvanovitch, Mileva. Equations de Gauss d'un sous-espace plongé dans l'espace Riemannien généralisé. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 615–621.

The author gives the equations of Gauss for the imbedding in the generalised Riemannian space with a nonsymmetric fundamental tensor $a_{\alpha\beta}$ introduced by Einstein in 1945 (not by Eisenhart in 1951). R. S. Mishra [Acad. Roy. Belg. Bull. Cl. Sci. (5) **40**, 1058–1071 (1954); MR **16**, 747] had already given the Mainardi-Codazzi equations for this case. The remark at the end of MR **16**, 747 holds also here [cf. Schouten, Ricci-calculus, 2nd. ed., Springer, Berlin, 1954, p. 184; MR **16**, 521].

J. A. Schouten (Epe).

Kron, Gabriel. A physical interpretation of the Riemann-Christoffel curvature tensor. The distribution of damping and synchronizing torques in oscillating transmission systems. Tensor (N.S.) **4** (1955), 150–172.

The author writes the equations describing the behavior of a rotating electrical machine as the equation of a path in an affine space with torsion in a non-holonomic coordinate system. The small oscillations of such a machine may be interpreted in terms of the relations existing between two neighboring paths. The paper is mainly concerned with the physical interpretation of the terms arising in the equations relating two neighboring paths.

A. H. Taub (Urbana, Ill.).

Moór, A., und Soós, Gy. Über affinzusammenhängende Mannigfaltigkeiten von Hyperflächenelementen, insbesondere deren Äquivalenz. Acta Sci. Math. Szeged **16** (1955), 29–42.

Let H_n be the manifold of hypersurface elements belonging to a differentiable point-manifold P_n. An element of H_n is the pair (x^i, u_i), where u_i is a hypersurface element at the point x^i. In H_n the authors assume the existence of an invariant differential of a vector density. This establishes H_n as an affine extension of the Cartan space in which H_n has a metric $L(x, u)$. The

authors develop the tensor calculus of such a space including covariant differentiation, curvature, and torsion. They also give necessary and sufficient conditions for the equivalence of two H_n's. *C. B. Allendoerfer.*

Galvani, O. Réalisations euclidiennes des plans de Finsler. Ann. Inst. Fourier, Grenoble **5** (1953–1954), 421–454 (1955).

The author considers the problem of realizing a two-dimensional, analytic Finsler space, F, by means of a variety W imbedded in a Euclidean or Riemannian space of three dimensions. The elements of W are the triples (M, Δ, P) where M is a point of F, Δ in a direction through M, and P is a plane through Δ.

The chief results are: if F is endowed with an absolute parallelism of linear elements, it can be realized in a Euclidean E^3; otherwise there exists a Riemannian V^3 in which F can be realized. The images of the points of F are the orthogonal trajectories of a one-parameter family of ruled developables of V^3 (or developables of E^3); the geodesics of F have the generators of these developables as their images. *C. B. Allendoerfer.*

Ispas, C. I. Identités de type Ricci dans l'espace de Finsler. Com. Acad. R. P. Romîne 2 (1952), 13–18. (Romanian. Russian and French summaries)

Starting from the covariant derivative in a Finsler space, the author establishes identities of Ricci type for the scalar S and for contravariant vectors V^i. These involve the connections of both E. Cartan and of L. Berwald as well as the second curvature tensor of the space. *C. B. Allendoerfer* (Seattle, Wash.).

Nagata, Yukiyoshi. Normal curvature of a vector field in a hypersurface in a Finsler space. Tensor (N.S.) **5** (1955), 17–22.

Let S be a hypersurface in a metric space X_n, C a curve on S. The covariant derivative with respect to the metric of X_n of a vector field tangential to S taken along C may be decomposed into normal and tangential components. The magnitude of the former component is called the normal curvature ${}_v\varkappa_n$ of the vector field with respect to C by T. K. Pan [Amer. J. Math. **14** (1952), 955–966; MR **14**, 406] who assumed that X_n is a Riemannian space. In the present paper the author generalises the results of Pan concerning this concept to the case that X_n is a Finsler space, basing his work on the theory of subspaces due to E. T. Davies [Proc. London Math. Soc. (2) **49** (1945), 19–39; MR **7**, 175], the element of support being the unit normal to S. The normal curvature $\varkappa_n$ of C appears as a special case of ${}_v\varkappa_n$, while the latter seems to possess several properties analogous to those of $\varkappa_n$. *H. Rund* (Toronto, Ont.).

Auslander, Louis. On curvature in Finsler geometry. Trans. Amer. Math. Soc. **79** (1955), 378–388.

The exact definitions are too involved to be given here, but they amount, very roughly, to this: In an n-dimensional Finsler space F, considered as a set of line elements (p, d) with a euclidean metric attached to each line element, let $(p, d; e_1, \cdots, e_n)$ be an orthogonal frame in the local euclidean geometry attached to (p, d). For a given vector field let e_n coincide with the field vector f. The euclidean metrics attached to $(p, f; e_1, \cdots, f)$ define a Riemannian metrization of the ground space R of the p. The space F is said to have curvature $\leqq k$ ($\geqq k$) if every Riemannian metric on R obtainable in this way has curvature $\geqq k$ ($\leqq k$). If for a curve C in R the unit tangent vector coincides with the field vector, then C has the same length in F and in the Riemannian metric defined by the field. A geodesic arc which is not a relative minimum for F, is not a relative minimum for a suitable Riemann metric, and conversely.

Hence, if the curvature of F is non-positive, there are no conjugate points, and the universal covering space of F (if complete) is homeomorphic to E^n. Whether any of the deeper properties of Riemann spaces with negative curvature carry over to F with negative curvature in the author's sense, is not discussed, nor is the relation to negative curvature in the sense of the reviewer or to that of F. P. Pedersen.

For complete F with non-negative curvature the following analogues to theorems of Myers and Synge are derived: If the curvature of F is greater than $k>0$, then F is compact and its diameter is at most $\pi k^{-\frac{1}{2}}$. A compact even-dimensional space F with non-negative curvature is simply connected. The paper reads "complete" instead of "compact", but the proof uses the "well-known" theorem that each free homotopy class contains a shortest geodesic. Similarly, Theorem 4.5 should read: "A compact (and not a complete) F with positive curvature has a finite fundamental group", because the proof uses that the curvature is greater than $k>0$. *H. Busemann.*

Kano, Chôtaro. On the crossed Jacobian extensor. Tensor (N.S.) **4** (1955), 173–181.

The purpose of this paper is to define the "crossed Jacobian extensor" and the "crossed Jacobian connection extensor" in connection with previous work of Craig [Amer. J. Math. **59** (1937), 764–774; Bull. Amer. Math. Soc. **53** (1947), 332–342; MR **8**, 491] and Craig and Guy [Amer. J. Math. **72** (1950), 229–246; MR **11**, 543]. On the basis of these definitions, Craig's intrinsic derivative is obtained in a new form. *C. B. Allendoerfer.*

Powell, F. C. A notation for vectors and tensors. Proc. Cambridge Philos. Soc. **51** (1955), 449–453.

An invariant equation between tensors depends only on the skeleton, that is the kernels, the places of the indices and the places where transvections act. It is often tried to give an index-free notation for this skeleton. The reasons why this never succeeded for valences >2 are elaborately dealt with in the reviewer's Einführung in die neueren Methoden der Differentialgeometrie [Bd. 1, 2nd ed., Noordhoff, Groningen-Batavia, 1935, p. 33 ff.] and Ricci-calculus [2nd ed., Springer, Berlin, 1954, pp. 58 ff.; MR **16**, 521]. The author gives us another example of such a "direct" calculus. It is certainly not worse than other inventions of this kind but the reviewer can only pass on the very good advice given to him by Felix Klein about 35 years ago: "Never use such stenography in your publications, otherwise nobody will read them." *J. A. Schouten* (Epe).

See also: Raševskiĭ, p. 124; Duschek und Hochrainer, p. 214.

Complex Manifolds

Frenkel, Jean. Sur les espaces fibrés analytiques complexes de fibre résoluble. C. R. Acad. Sci. Paris **241** (1955), 16–18.

This generalises some previous results of the author [same C. R. **236** (1953), 40–41; MR **14**, 573]. Let I, Λ be

two finite sets (maybe void) of indices, D a Stein manifold and

$$X=D\times(\prod_{i\in I}X_i)\times(\prod_{\lambda\in\Lambda}Y_\lambda),\quad X_i=C^{d(i)+1}-\{0\},$$

where $C^{d(i)+1}$ denotes the complex affine space of dimension $d(i)+1$ and Y_λ the complex projective space of dimension $d(\lambda)$. Suppose V_m to be the sheaf of germs of a complex linear space bundle over X with a connected solvable complex Lie group as the structural group. The author first proves that $H^p(X, V_m)=0$ when all $d(i)\neq 1$, $p\neq 0$, and $p\neq\sum_{i\in J}d(i)+\sum_{\lambda\in\Gamma}d(\lambda)$ for any subsets J, Γ of I, Λ respectively. Then the author uses this to discuss the complex analytic principal fibre spaces E over X with a solvable complex Lie group G as the structural group. Among other results of similar nature, the following are proved. (A) Suppose that all the $d(i)$ are different from 1. Then the topological triviality of E implies the complex analytic triviality. If, moreover, G is nilpotent, then the topological X-isomorphism of two such fibre spaces E implies the complex analytic X-isomorphism. (B) Suppose that all the $d(i)>2$, all the $d(\lambda)\neq 2$, and at most one of the $d(\lambda)$ is equal to 1. Then every principal fibre space (in the topological sense) over X with a connected solvable complex Lie group as the structural group has a complex structure compatible with the fibre structure. *H. C. Wang* (Seattle, Wash.).

Lelong, P. Prolongement analytique et singularités complexes des fonctions harmoniques. Bull. Soc. Math. Belg. **1954**, 10–23 (1955).

Let R^n $(x_j'=0)$ be the real subspace of the complex space C^n $(X_j=x_j+ix_j')$. A real analytic function $f(x)$ in R^n, with domain of maximal real analyticity $d\subset R^n$, can be extended to a holomorphic function $f(X)$ in C^n with domain of holomorphy Δ. The author gives a summary of some investigations concerning the relationship between d and Δ, especially in the case that $f(x)$ is a harmonic function. For each $t\in R^n$ let $\Gamma(t)$ be the cone in C^n defined by $\sum_k(X_k-t_k)^2=0$; the cell of harmonicity $H(d)$ of d is that connected component of the set $C^n-\sum_t\Gamma(t)$ which contains d, where t runs through the boundary of d. Any function harmonic and univalent in d can be extended to a function holomorphic in $H(d)$ but not necessarily univalent there. If $n=2$ and d is the domain of real analyticity of a harmonic function $f(x)$, then $H(d)$ is the domain of holomorphy of its extension $f(X)$; this is false when $n\geqq 3$. Nevertheless there is always a function harmonic in d having $H(d)$ as its domain of holomorphy. An alternative definition of the cell of harmonicity is given in terms of a dual cone, which permits an extension of this notion and of some of the results to cases in which d is no longer univalent. Some applications are also given. *R. C. Gunning* (Chicago, Ill.).

Hua, Loo-Keng. On the theory of functions of several complex variables. I. A complete orthonormal system in the hyperbolic space of matrices. J. Chinese Math. Soc. 2 (1953), 288–323. (Chinese. English summary)

Let R_{mn} be the "hyperbolic space" of all $m\times n$ matrices $Z=Z^{(m,n)}=(z_{jk})$ with complex elements satisfying $I-Z\bar{Z}'>0$; here $I=I^{(m,m)}$ is the $m\times m$ unit matrix, and the inequality means that the Hermitean matrix $I-Z\bar{Z}'$ is positive definite. Put $z_{jk}=x_{jk}+iy_{jk}$ with real x_{jk}, y_{jk}, and $\dot{Z}=\prod_{j=1}^m\prod_{k=1}^n(dx_{jk}dy_{jk})$. The analytic functions $f(Z)$ of the mn variables z_{jk} satisfying

$$\iint\cdots\int_{I-Z\bar{Z}'>0}|f(Z)|^2\dot{Z}<\infty$$

form a set B^2 which by a general existence theorem of S. Bergmann [Sur les fonctions orthogonales de plusieurs variables complexes. ⋯, Interscience, New York, 1941; MR 2, 359] possesses an orthonormal system. In the present paper, the author determines this system explicitly. First let $m=n$, and let $A_1(X)$, $A_2(X)$, $\cdots$ be all the inequivalent irreducible rational integral representations of the general linear group, X being any $n\times n$ matrix, while $A_i(X)=(a_{jk}^{(i)}(X))$ is, say, an $a_i\times a_i$ matrix with polynomial elements. By Schur's lemma [e.g., H. Weyl, The classical groups, Princeton, 1939, p. 79; MR 1, 42], it is easily shown that these polynomials are already orthogonal in B^2, and that further

$$\iint\cdots\int_{I-Z\bar{Z}'>0}|a_{jk}^{(i)}(Z)|^2\dot{Z}=\varrho_i>0$$

is independent of j and k. Hence, if $\chi_i(X)$ denotes the character of the representation $A_i(X)$, then also

$$\iint\cdots\int_{I-Z\bar{Z}'>0}\chi_i(Z\bar{Z}')\dot{Z}=a_i^2\varrho_i, \tag{1}$$

and the modified polynomials $\varphi_{jk}^{(i)}(Z)=(1/\sqrt{\varrho_i})a_{jk}^{(i)}(Z)$ are moreover normed in B^2. The evaluation of ϱ_i by means of (1) forms the main difficulty, which the author solves in a very elegant manner. He shows, by means of several changes of variables, that

$$a_i^2\varrho_i=2^{-n}\omega_n\omega_n'\int_0^1\cdots\int_0^1\chi_i(\Lambda)\Delta(\lambda_1,\cdots,\lambda_n)^2d\lambda_1\cdots d\lambda_n.$$

Here Λ is a diagonal matrix $[\lambda_1,\cdots,\lambda_n]$ with $\lambda_j>0$; $\Delta(\lambda_1,\cdots,\lambda_n)=\prod_{j>k}(\lambda_j-\lambda_k)$; and ω_n, ω_n', which are defined as the volumes of the unitary group and one of its quotient groups, have the product

$$\omega_n\omega_n'=2^n\pi^{n^2}/(n!\{1!2!\cdots(n-1)!\}^2).$$

To evaluate the integral over $\lambda_1,\cdots,\lambda_n$ the author introduces the signature $(f_1,f_2,\cdots,f_n)$, $f_1\geqq f_2\cdots\geqq f_n\geqq 0$, of the character $\chi_i(X)$. Let $l_j=f_j+n-j$, $D(x_1,\cdots,x_n)=\prod_{j>k}(x_j-x_k)$, and $|\lambda^{l_1},\lambda^{l_2},\cdots,\lambda^{l_n}|=|a_{ij}|$ with $a_{ij}=\lambda_i^{l_j}$ $(i,j=1,\cdots,n)$. Then

$$a_i=N(f_1,\cdots,f_n)=D(l_1,\cdots,l_n)/D(n-1,n-2,\cdots,0)$$

and for diagonal matrices Λ,

$$\chi_i(\Lambda)=|\lambda^{l_1},\cdots,\lambda^{l_n}|/\Delta(\lambda_1,\cdots,\lambda_n),$$

and from this ϱ_i takes the final form

$$\varrho_i=\frac{\pi^{n^2}}{N(f_1,\cdots,f_n)}\prod_{j=1}^n\frac{l_j!}{(n+l_j)!}.$$

It is further found that the generating function

$$\sum_{i=1}^\infty\sum_{j,k}\varphi_{jk}^{(i)}(Z)\varphi_{jk}^{(i)}(\bar{W}')=c(I-Z\bar{W}')^{-2n}$$

with

$$c^{-1}=\iint\cdots\int_{I-Z\bar{Z}'>0}\dot{Z}=\frac{\pi^{n^2}(1!2!\cdots(n-1)!)^2}{1!2!\cdots(2n-1)!}.$$

Finally the restriction $m=n$ is got rid of, but it suffices to consider the case when $m\leqq n$. By adding zero elements, Z is enlarged to an $n\times n$ matrix, and explicit formulae are derived from those for $m=n$. A proof is given that also now the full system $\{\varphi_{jk}^{(i)}(Z)\}$ is complete. The paper ends with a generalized Cauchy integral formula for $f(Z)$. *K. Mahler* (Zbl 52, 311).

See also: Ozaki, Kashiwagi, and Tsuboi, p. 177; Arens and Calderón, p. 177.

Algebraic Geometry

de Tullio Cirillo, Elda. La trasformazione birazionale (2,7) dell' S_{11} complesso immagine di una trasformazione cremoniana quadratica dell' S_3 complesso prolungata nel campo tripotenziale. Giorn. Mat. Battaglini (5) 3(83) (1955), 83–88.

Balsimelli, Pio. Su una trasformazione birazionale dell'S_3 biduale. Giorn. Mat. Battaglini (5) 3(83) (1955), 77–81.

Manara, Carlo Felice. I gruppi ciclici di trasformazioni piane di Jonquières. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)** (1954), 115–129.

Les transformations T cycliques d'ordre n de de Jonquières peuvent se classer selon les types suivants. 1°. Celles qui dans le faisceau invariant de droites subordonnent l'identité: a) si $n\geqq 2$ ce sont des homologies cycliques d'ordre n et on peut leur donner la forme canonique

$$x'=x,\quad \frac{a(x')y'+b(x')}{c(x')y'+d(x')}=\varepsilon\frac{a(x)y+b(x)}{c(x)y+d(x)}\quad (\varepsilon^n=1),$$

a, b, c, d étant des polynômes, la courbe des points unis est alors décomposée en deux courbes rationnelles; b) $n=2$, transformation $x'=x$, $yy'=F(x)$, F étant un polynôme; la courbe des points unis est alors hyperelliptique irréductible. 2°. Celles qui dans le faisceau invariant de droites subordonnent une projectivité d'ordre n; elles laissent invariant un faisceau de courbes rationnelles unisécantes des droites du faisceau, par lequel on peut les réduire au type 1°. 3°. Celles qui dans le faisceau invariant de droites subordonnent une projectivité de période s diviseur de n [cas qui avait échappé à T. Turri, Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 33–43; MR **11**, 686]: a) si $2s<n$, la transformation est birationnellement équivalente à une du type 1°, car il existe un faisceau invariant de courbes rationnelles. b) $n=2s$ alors T^s a une courbe irréductible de points unis si s est impair; on peut ramener T à la forme canonique $x'=\eta x$, $yy'=F(x)$, $\eta^s=1$, F polynôme invariant par $x'=\eta x$; si la courbe $y^2=F(x)$ est non rationnelle, cette transformation ne peut se ramener au type 1°. *B. d'Orgeval* (Dijon).

Bilo, J. Sur les cubiques gauches et sur les surfaces cubiques ayant quatre points doubles. Mathesis **64** (1955), 93–97.

Godeaux, Lucien. Sur les points de diramation de seconde espèce et de première catégorie d'une surface multiple. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 703–708.

Godeaux, Lucien. Sur la construction d'exemples de surfaces algébriques contenant des involutions cycliques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 798–804.

Zobel, A. A note on the variety of Gherardelli. Quart. J. Math. Oxford Ser. (2) **6** (1955), 143–146.

In this note the author deals with the irreducible nonsingular variety W whose points represent unexceptionally the ∞^4 second-order curve elements E_2 in the plane as defined by F. Engel and E. Study: this is the variety first constructed by G. Gherardelli [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2** (1941), 821–828; MR **8**, 222; see also J. G. Semple, Proc. London Math. Soc. (3) **4** (1954), 24–49; MR **15**, 820].

Algebraic bases for curves and threefolds on W were found by Gherardelli. Using the method of degenerate collineations as in van der Waerden [Math. Ann. **115** (1938), 645–655] and Hodge and Pedoe [Methods of algebraic geometry, v. II, Cambridge, 1952; MR **13**, 972], the author finds in this note a base for surfaces on W, showing also that the base he thus found is minimal in Severi's sense. *F. Gherardelli* (Florence).

Scafati, Maria. Sulle superficie ellittiche con un fascio ellittico di curve di genere quattro. Rend. Mat. e Appl. (5) **14** (1955), 289–337.

In questa Memoria l'A. espone dettagliatamente i risultati ottenuti nella classificazione delle superficie ellittiche, già riassunti in una precedente Nota [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16** (1954), 721–724; MR **16**, 741]. Come è noto, une superficie ellittica F possiede un fascio ellittico $\{C\}$ di curve di un certo genere π ed un fascio $\{K\}$ di curve ellittiche di genere eguale al genere geometrico p_g della superficie. Se con $n=[C, K]$, t, s_i si indicano rispettivamente il determinante della superficie F, il numero delle curve multiple del fascio $\{C\}$ e le loro molteplicità, per la formula di Zeuthen si ha:

$$(1)\qquad n(2p_g-2)+\sum_{i=1}^{t}\frac{n}{s_i}(s_i-1)=2\pi-2.$$

Nella presente Memorie si determinano dapprima le soluzioni aritmetiche della (1), quando si ponga in essa $\pi=4$; nella seconda parte della Memorie si prova l'effettiva esistenza e si costruiscono le superficie ellittiche corrispondenti ai casi in cui le curve C di genere 4 sono iperellittiche. Questo scopo viene raggiunto attraverso la classificazione birazionale delle curve iperellittiche di genere 4 con un gruppo di trasformazioni birazionali in sè, ciclico o abeliano a base due. *F. Gherardelli* (Firenze).

Panella, Gianfranco. Intorno ad alcune superficie ellittiche. Rend. Mat. e Appl. (5) **14** (1955), 342–354.

Nel presente lavoro, partendo da risultati di E. Ciani [Rend. Circ. Mat. Palermo **36** (1913), 58–78] sulle quintiche piane autoproiettive, si classificano le quintiche piane di genere $\pi>3$ con un gruppo di proiettività ciclico o abeliano a base due. I risultati di questa classificazione permettano all'A. di determinare le superficie ellittiche F, di genere geometrico nullo possedenti un fascio ellittico $\{C\}$ di genere $\pi=4, 5, 6$ sotto l'ipotesi che il determinante n di F eguagli l'ordine del gruppo completo di proiettività spettante ad una quintica piana autoproiettiva di genere π. I risultati cosi ottenuti vengono utilmente confrontati con le soluzioni aritmeticamente possibili del problema, che si ottengono discutendo l'equazione di analisi diofantea:

$$2\pi-2=n(2p_g-2)+\sum_{i=1}^{t}\frac{n}{s_i}(s_i-1).$$

[per le notazioni qui usate cfr. la recensione precedente]. *F. Gherardelli* (Firenze).

Malesani, Paolo. Su un criterio per l'esistenza di varietà unisecanti. Ann. Univ. Ferrara. Sez. VII. (N.S.) 3 (1954), 55–66.

In questa Nota l'A. trova una condizione sufficiente affinchè si possa costruire una varietà unisecante per un sistema algebrico ∞^t di varietà algebriche tale che la generica V_r^n contenga una varietà s-pla W_{r-1}, completa intersezione della V_r^n con un'altra varietà ad r dimensioni.

Se la V_r^n e la W_{r-1} soddisfano a una conviente ipotesi di generalità, la condizione, data dall'A., è anche sufficiente.
M. Piazzolla Beloch (Ferrara).

Barsotti, Iacopo. Structure theorems for group-varieties. Ann. Mat. Pura Appl. (4) **38** (1955), 77–119.

Ce mémoire donne, sur la structure des groupes algébriques commutatifs, des résultats qui ont été trouvés indépendemment par l'auteur, par C. Chevalley et par M. Rosenlicht. Ici un groupe algébrique (ou variété de groupe) désigne un ouvert G (au sens de la topologie de Zariski) d'une variété projective, muni d'une loi de composition et de l'application $x \rightarrow x^{-1}$, de sorte que ce soient des applications partout régulières de $G \times G$ dans G et de G dans G, et que les axiomes algébriques usuels soient vérifiés. La restriction aux ouverts de variétés projectives est d'ailleurs inessentielle.

Une section préliminaire montre que les théorèmes d'isomorphisme de la théorie des groupes s'étendent à la structure plus riche de groupe algébrique, modulo inséparabilité. L'étude des groupes algébriques commutatifs est basée sur les résultats suivants: un groupe algébrique commutatif simple de dimension >1 est une variété abélienne; la généralisation du théorème de complète réductibilité de Poincaré au cas où le groupe ambient est un groupe algébrique commutatif quelconque; et l'extension de la notion de système de facteurs (et donc de produit croisé) à la structure de groupe algébrique commutatif. Ceci étant, tout groupe algébrique G contient un plus grand sous groupe rationnel B, G/B est une variété abélienne, et G est produit croisé de G/B et B. En caractéristique 0 un group algébrique commutatif rationnel B est un produit de droites additives ou multiplicatives. En caractéristique $p \neq 0$ s'introduisent aussi des groupes algébriques de torsion (dits ,,périodiques") qui sont des produits croisés de droites additives; un groupe algébrique commutatif rationnel B est produit direct de son plus grand sous groupe périodique, et d'un produit de droites multiplicatives.

L'étude de l'équivalence algébrique sur une variété abélienne donne une esquisse (incomplète) de la démonstration du fait suivant: le groupe des classes d'extensions d'un groupe algébrique commutatif par un autre a une structure naturelle de groupe algébriques. On donne quelques résultats élémentaires sur les dérivations invariantes du corps des fonctions sur un groupe algébrique. L'auteur aborde ensuite le cas non commutatif et montre que le quotient G/C d'un groupe algébrique G par son centre C est un sous groupe algébrique d'un groupe linéaire (un ,,groupe de Vessiot"); il en déduit quelques renseignements sur la structure de G; en particulier toute sous variété abélienne de G est contenue dans le centre C de G.

Des remarques finales s'occupent des formes différentielles invariantes, et de la méthode des Algèbres de Lie (insuffisante en caractéristique $p \neq 0$). Des exemples variés illustrent les phénomènes particuliers à la caractéristique $p \neq 0$. *P. Samuel* (Clermont-Ferrand).

Chow, Wei-Liang. On Abelian varieties over function fields. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 582–586.

Dans un mémoire récent [Trans. Amer. Math. Soc. **78** (1955), 253–275; MR **16**, 743; nous renvoyons le lecteur à cette analyse pour la terminologie et les notations], l'auteur a montré que, étant données une variété abélienne $A°$ définie sur une extension régulière $K°$ d'un corps k, et le monomorphisme $F': A' \rightarrow A°$ de la k-trace A' de $A°$ dans $A°$, alors, pour m suffisament grand, l'application ,,diagonale" $F_{(m)}'$ de A' dans le produit $A_{(m)}°$ de m spécialisations génériques $A_i°$ de $A°$ est un monomorphisme régulier (c.à.d. tel que $K'(y)$ soit extension régulière de $K'(F_{(m)}'(y))$, K' désignant un corps de définition de $F_{(m)}'$ et y un point générique de A' sur K'). On démontre ici la propriété parallèle de la k-image A de $A°$: en notant $F: A° \rightarrow A$ l'épimorphisme canonique de $A°$ sur A, alors l'épimorphisme $F_{(m)}$ de $A_{(m)}°$ sur A défini par

$$F_{(m)}(x_1, \cdots, x_m) = \sum_i F_i(x_i)$$

($(x_1, \cdots, x_m)$ point générique de $A_{(m)}°$, F_i épimorphisme $A_i° \rightarrow A$ transporté de F) est régulier pour m grand. Dans la démonstration, le passage par des variétés abéliennes quotients des $A_{(m)}°$ amène d'abord à l'étude d'une extension algébrique, qu'on montre être triviale pour m grand; la technique familière à l'auteur, qui consiste à ,,doubler" la variété abélienne $A_{(m)}°$ donne alors le résultat. Comme conséquence on montre que la k-image de $A°$ est invariante par extension k' du corps de base k (pourvu, bien sûr, que k' et $K°$ soient algébriquement disjointes).
P. Samuel (Clermont-Ferrand).

Abhyankar, Shreeram. On the ramification of algebraic functions. Amer. J. Math. **77** (1955), 575–592.

Let K denote an r-dimensional field of algebraic functions over an algebraically closed ground field k of characteristic p, and let K^* be a finite separable extension of K. Let V and V^* be normal models of K/k and K^*/k such that the transformation T of V^* onto V determined by the embedding $K \subset K^*$ and its inverse T^{-1} are free of fundamental points. Let W^* be an irreducible subvariety of V^* and let $W = T(W^*)$. Let R and R^* be the quotient rings at W and W^*, let E and E^* be the quotient fields of the completions of R and R^*, and let F and F^* be the residue fields of R and R^*. The ramification index $r(W^*:W)$ of W^* over W is defined to be $[E^*:E]\cdot[F^*:F]_s^{-1}$, and W is a branch variety for T if $r(W^*:W) > 1$. The branch locus $B(T)$ is the set of all rational points P of V/K such that P is on some irreducible subvariety of V/k which is a branch variety for T. $B(T)$ is an algebraic subvariety of V and $W \subset B(T)$ is a necessary and sufficient condition for an irreducible variety W to be a branch variety.

When W is an irreducible simple subvariety of V and $r(W^*:W) > 1$, there is an $(r-1)$-dimensional irreducible subvariety U^* of V^* which contains W^* and is such that $r(U^*:T(U^*)) > 1$. From this it follows that the components of $T^{-1}\{B\}$ and B through W^* and W are $(r-1)$-dimensional, and that if $B(T)$ contains no singular point of V then $B(T)$ is purely $(r-1)$-dimensional. This theorem is due to Zariski, and the author ascribes the proof given in the present paper to him.

When W and W^* are points P and P^* of V and V^*, let H^* be the least Galois extension of E which contains E^*, and let $G(P^*/P)$ denote the Galois group of H^* over E. Let π be the least subgroup of $G(P^*/P)$ which contains all of the p-Sylow groups of $G(P^*/P)$. (Take $\pi = (1)$ if $p = 0$.) One of the main results of the paper is to the effect that if P is a simple point of V and if the component of B through P which is ramified for T has a t-fold normal crossing at P, then $G(P^*/P)/\pi$ is a direct product of s $(\leq t)$ cyclic subgroups. The term "t-fold normal crossing" signifies that the ideal of B in the local ring at P splits into T components each of which is principal, and the generators of these principal ideals together form part of a minimal local basis for the ideal of P. Thus in particular, when $r = 2$, $p = 0$, and P is a simple point of B, the group

$G(P^*/P)$ is cyclic, and if P is an ordinary double point of B then $G(P^*/P)$ is a direct product of two cyclic groups. These results (when $r=2$ and $p=0$) were known. Indeed, the paper contains a critical analysis of Jung's proof of the local uniformization theorem which shows that they are key facts upon which this proof is based. Examples are given to show that they fail to hold when $p\neq 0$, so that Jung's methods are not adaptable to the case of modular fields. *H. T. Muhly.*

Denniston, R. H. F. On the topology of the joined point-pairs of an algebraic variety. Ann. Mat. Pura Appl. (4) **38** (1955), 213–223.

An ordered point-pair, in a given n-dimensional complex projective space S, consists of a pair of points taken in a definite order. A joined point-pair consists of an ordered point-pair (P, Q) together with a line R which contains P and Q. The author defines joined point-pairs of an irreducible non-singular algebraic variety V in S, and remarks, very justly, that this concept is of importance in both classical and modern algebraic geometry. V is taken to be a rational variety, subject to certain restrictions. It is then proved that the set of joined point-pairs of V is a fibre-bundle. This theorem makes it possible for the author to determine the homology-groups, and also a base for each group. It is noted that, although the restrictions on the varieties studied are severe, quadric and Grassmann varieties are included amongst them. *D. Pedoe* (Khartoum).

NUMERICAL ANALYSIS

Camp, Kingsland. New possibilities in graduation. Soc. Actuar. Trans. 7 (1955), no. 3, 6–30.

This article describes a suggested program for performing on an electronic computer a graduation of a set of mortality rates by the Whittaker-Henderson "B" method [Henderson, Trans. Actuar. Soc. Amer. **26** (1925), 52–57]. *T. N. E. Greville* (Washington, D.C.).

Ionescu, D. V. Le reste dans la formule "préférée" de Newton. Gaz. Mat. Fiz. Ser. A. 7 (1955), 298–302. (Romanian. Russian and French summaries)

Spenser, Gordon. Random numbers and their generation. Computers and Automation 4 (1955), no. 3, 10–11, 23.

Clenshaw, C. W. A note on the summation of Chebyshev series. Math. Tables Aids Comput. 9 (1955), 118–120.

Calvo Carbonell, Carlos. A series which gives the root of an algebraic or transcendental equation. Rev. Acad. Ci. Madrid **48** (1954), 189–201. (Spanish)

Let $y=f(x)$ be a function with derivatives of all orders at a point x_0. Here is discussed the known formula for the inverse function expanded in a Taylor series

$$x=F(y)=x_0+(y-y_0)/f'(x_0)$$
$$+(y-y_0)^2(-f''(x_0))/2f'^3(x_0)+\cdots$$

in order to calculate the zeros of $f(x)$. This formula is developed in Willers [Practical analysis, Dover, New York, 1948, pp. 222–223; MR **10**, 404]. In this paper examples are given which illustrate the method, as well as the error involved in stopping with a certain term of the series. *E. Frank* (Chicago, Ill.).

★ **Faddiejewa, W. N. Metody numeryczne algebry liniowej. [Numerical methods of linear algebra.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1955. 208 pp. zł 17.90.

Translation into Polish by A. Siemionow of Faddeeva's Vyčislitel'nye metody lineĭnoĭ algebry [Gostehizdat, Moscow, 1950; MR **13**, 872].

Chao, F. H. A difference equation method for solving simultaneous equations. Acta Math. Sinica 5 (1955), 149–159. (Chinese. English summary)

The author interprets the solution of a system of linear equations as the solution of a system of linear difference equations taking on a number of zero initial values and a number of zero final values. A systematic computational arrangement for solving systems of linear equations is given. Actually the method discussed is an elimination method. *Ky Fan* (Notre Dame, Ind.).

Chao, Robert F. H. A gradient method for solving simultaneous equations. Acta Math. Sinica 3 (1953), 328–342. (Chinese. English summary)

This is an expository article on the gradient method for solving systems of linear equations, for finding complex zeros of polynomials and for finding quadratic factors of quartic polynomials. *Ky Fan* (Notre Dame, Ind.).

Greenspan, Donald. Methods of matrix inversion. Amer. Math. Monthly **62** (1955), 303–318.

This is an elementary description of eleven known methods for inverting a finite matrix, illustrated by numerical examples of order 3. There is a bibliography of 34 titles. *G. E. Forsythe* (Los Angeles, Calif.).

Brenner, J. L., and Reitwiesner, G. W. Remark on determination of characteristic roots by iteration. Math. Tables Aids Comput. 9 (1955), 117–118.

Bellman, Richard. On an iterative procedure for obtaining the Perron root of a positive matrix. Proc. Amer. Math. Soc. 6 (1955), 719–725.

The author gives a new iterative procedure for computing the unique maximum characteristic root $p(A)$ of a positive matrix A. The algorithm is based on the fact that $p(A)$ can be alternatively defined by

$$p(A)=\max_x \min_i \sum_{j=1}^n a_{ij}x_j/x_i=\min_x \max_i \sum_{j=1}^n a_{ij}x_j/x_i,$$

where $x=(x_i, \cdots x_n)$ satisfies $x_j \geq \min_{i,j} a_{ij}/\max_i (\sum_j a_{ij})$. Using this fact, it is shown that the recurrence formula $u_{n+1}=\min_x \max_i [\sum_{j=1}^n a_{ij}x_j+u_n(1-x_i)]$ gives a sequence of numbers u_n converging to $p(A)$. *D. Gale.*

Gagua, M. On convergence of Galerkin's method. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 665–668. (Russian)

Let X be a normed space with a complete set of elements x_i, Y a subspace of Hilbert space, K a mapping of X on Y. Let X_n be the subspace of X defined by $x_1, \cdots, x_n$; Y_n the subspace of Y defined by $Kx_1, \cdots, Kx_n$. Let

$$E_n(x)=\inf_{x_n\in X_n} \|x-x_n\|, \quad E_n(y)=\inf_{y_n\in Y_n} \|y-y_n\|.$$

Let elements $\psi_i \in Y$ form a complete orthonormal system for Y with

$$\Delta_n = \begin{vmatrix} (Kx_1, \psi_1) \cdots K(x_1, \psi_n) \\ \cdots \\ (Kx_n, \psi_1) \cdots K(x_n, \psi_n) \end{vmatrix} \neq 0.$$

Then if

$$\sum_1^n (Kx_i, \psi_j) c_i^* = (y, \psi_j),\ \bar{x}_n^* = \sum_1^n c_i^* x_i,$$

and

$$\lim_n \Delta_n^{-1} E_n(y) \Big[\sum_1^n \|x_i\|^2 \sum_1^n \|\psi_i\|^2 \Big]^{\frac{1}{2}} = 0,$$

it follows that

$$\|x - \bar{x}_n^*\| = O\Big\{ E_n(x) + |\Delta_n^{-1}| E_n(y) \Big[\sum_1^n \|x_i\|^2 \sum_1^n \|\psi_i\|^2 \Big]^{\frac{1}{2}} \Big\}.$$

This is Theorem 2, obtained as a consequence of a somewhat more general Theorem 1. *A. S. Householder.*

Gagua, M. On approximate solution of linear boundary problems for elliptic equations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1061–1064. (Russian)

This paper applies the method of Galerkin as expounded in the paper reviewed above. *A. S. Householder.*

Slobodyanskiĭ, M. G. Estimates of errors of approximate solutions of linear problems. Prikl. Mat. Meh. **17** (1953), 229–244. (Russian)

In many linear boundary-value problems, the determination of the unknown function, or of its derivatives up to a certain order, or of a linear operator of the unknown function, may be shown to be equivalent to the determination of the scalar product (f, v), where $Au = f$, $Av = \varphi$, with f and φ given elements of a real Hilbert space H, and u and v unknown elements of the linear manifold of H where the positive-definite, symmetric, linear operator A is defined. The author has given earlier a method for the construction of numerical approximations to (f, v) under such circumstances [Prikl. Mat. Meh. **16** (1952), 449–464; MR **14**, 502]. The present paper contains further developments of this method, and numerical applications to boundary-value problems for ordinary differential equations, e.g.

$$Au \equiv -\frac{d}{dx}\frac{du}{dx} + (1+x)u = 1\ (0 < x < 1),$$

with boundary conditions $u(0) = u(1) = 0$.

J. B. Diaz (College Park, Md.).

Slobodyanskiĭ, M. G. Approximate solution of some boundary problems for elliptic differential equations and estimates of the error. Dokl. Akad. Nauk SSSR (N.S.) 89 (1953), 221–224. (Russian)

K. O. Friedrichs [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929,** 13–20] has given a method for transforming the probiem of minimizing certain integrals into the problem of maximizing certain other expressions, and has worked out some special cases when several independent variables occur. The author considers the boundary-value problem consisting of the equation

$$Au = \sum_{j=1}^m \sum_{k=1}^m \frac{\partial}{\partial x_j}\Big(p_{jk}\frac{\partial u}{\partial x_k}\Big) + cu = f, \text{ in } \Omega,$$

subject to the boundary conditions $u|_s = 0$, or

$$\Big[\sum_{j=1}^m \cos(v, x_j) \sum_{k=1}^m p_{jk} \frac{\partial u}{\partial x_k} \Big]_s = 0$$

on the boundary S of Ω. Under suitable hypotheses on the coefficients of the equation, the operator A is positive definite and the boundary-value problems considered lead to the minimization of the integral

$$F_f^A(u) \equiv \int_\Omega \Big\{ \sum_{j=1}^m \sum_{k=1}^m p_{jk} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k} + cu^2 \Big\} d\Omega - 2\int_\Omega fu\, d\Omega.$$

In order to be able to apply to this problem a method previously developed [Prikl. Mat. Meh. **16** (1952), 449–464, MR **14,** 502], the author is led to construct (cf. reference to Friedrichs above) a functional whose maximum is the minimum of F_f^A. *J. B. Diaz.*

Slobodyanskiĭ, M. G. On approximate solution of linear problems reducing to variational ones. Prikl. Mat. Meh. 17 (1953), 623–626. (Russian)

In the two papers reviewed above the author has been concerned with the question of constructing functionals whose maximum is the minimum of the functional [cf. K. O. Friedrichs, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929**, 13–20]

$$\int_\Omega F(u, u_x, \cdots)\, d\Omega - 2\int_\Omega fu\, d\Omega\ \Big(u_x \equiv \frac{\partial u}{\partial x}, \cdots\Big),$$

where $F(u, u_x, \cdots)$ is a positive definite quadratic form in the variables $u, u_x, \cdots$. The present paper considers the case when F need not be a positive definite quadratic form, and an application is made to the following boundary-value problem:

$$\sum_{k=0}^m (-1)^k \frac{d^k}{dx^k}\Big[p_k(x)\frac{d^k u}{dx^k}\Big] = f(x);\ a < x < b;$$

$$u(a) = \cdots = u^{(m-1)}(a) = 0,\ u(b) = \cdots = u^{(m-1)}(b) = 0,$$

where the coefficients p_k may change sign.

J. B. Diaz (College Park, Md.).

Slobodyanskiĭ, M. G. On transformation of the problem of the minimum of a functional to the problem of the maximum. Dokl. Akad. Nauk SSSR (N.S.) 91 (1953), 733–736. (Russian)

The author presents a refinement of a method given previously [Prikl. Mat. Meh. **16** (1952), 449–464; MR **14,** 502] for the numerical approximation of the scalar product (f, v), where $Au = f$, $Av = \varphi$, with f and φ given elements of a real Hilbert space H, and u and v unknown elements of the linear manifold of H where the positive-definite, symmetric, linear operator A is defined. By way of an application, bounds are found for the absolute value of the solution of the boundary-value problem

$$-\frac{d}{dx}\Big(p_1\frac{du}{dx}\Big) + p_0 u = f(x)\ (0 < x < 1),\ u(0) = u(1) = 0,$$

where $p_1(x) \geq c_0 > 0$, and $p_0(x) \geq 0$ on the interval $0 \leq x \leq 1$.

J. B. Diaz (College Park, Md.).

Svirskiĭ, I. V. On the construction of variational methods of computation. Prikl. Mat. Meh. 19 (1955), 453–462. (Russian)

Given a system of equations $A\varphi = 0$, where A represents a nonlinear operator corresponding to the conditions to be satisfied within and on the boundaries of a region of interest, suppose it required to evaluate a functional $\Phi(\varphi)$, but not necessarily to determine φ itself. An example is the determination of an eigenvalue. The author describes a technique for transforming to a variational problem, the method of Galerkin being a special case, and illustrates with several special problems. It is assumed that the function φ_0 is known for some

simpler region approximating the one in question and the method is of interest by virtue of the fact that a function is not sensitive to its argument near an extremum. *A. S. Householder* (Oak Ridge, Tenn.).

Gray, H. J., Jr. Propagation of truncation errors in the numerical solution of ordinary differential equations by repeated closures. J. Assoc. Comput. Mach. 2 (1955), 5–17.

The numerical integration of ordinary differential equations can be conveniently based on an extrapolation formula $f(t_n)=\sum_{j=1}^{N} a_j f(t_{n-j})+h\sum_{j=0}^{M-1} b_j f'(t_{n-j})$, since one can readily accumulate the values of f and its derivative f' for values of t which differ by multiples of h. For a system of linear differential equations with constant coefficients, the author points out that the essential effect of using such a numerical process instead of ordinary integration is to replace the exponential functions exp (λt) in the solution by functions exp (μt), where λ and μ are related by a transcendental equation. The author indicates that the accuracy of the solution can be referred to the accuracy to which $-\log(1-s)$ as a function of the complex variable s can be approximated by P/D, where $P=1-\sum_{j=1}^{N} a_j(1-s)^j$ and $D=\sum_{j=0}^{M-1} b_j(1-s)^j$. In particular, by means of the "Padé approximants" the coefficients a_j and b_j can be chosen so that the approximation to log $(1-s)$ is of the $(N+M)$-th order. The author obtains the precise analytic relation between $z=\lambda h$ and $w=\mu h$. The error is associated with a power series in s. Let gs^{N+M} be the first non-zero term of this error series. The author gives a table of the values of g which clearly indicates that the most accurate solution method would be one in which $N\cong M$. The author however points that these methods are "usually unstable", although the cases $N=M=1$ and $N=M=2$ are stable. The author lists the coefficients a_i and b_i for a number of integration methods with $N+M\leqq 8$.
F. J. Murray (New York, N.Y.).

Sokolov, Yu. D. On a method of approximate solution of linear integral and differential equations. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 107–111. (Ukrainian. Russian summary)

This paper presents a new method for the approximate solution of linear integral and differential equations, different from the usual integrating process. The convergence of the method is proved and an estimate of the error established. Two examples are given applying the method to a one-dimensional boundary problem for a linear differential equation of second order and one example applying it to the case of a nonlinear equation of the second order. *W. E. Milne* (Corvallis, Ore.).

Bnd order

★**Panow, D. J. Formelsammlung zur numerischen Behandlung partieller Differentialgleichungen nach dem Differenzenverfahren.** Akademie-Verlag, Berlin, 1955. x+130 pp. DM 12.00.

Translation by K. Borkmann and W. Schulz of Panov's Spravočnik po čislennomu rešeniyu differencial'nyh uravnenil v častnyh proizvodnyh [Gostehizdat, Moscow, 1951; MR **14**, 93].

Peaceman, D. W., and Rachford, H. H., Jr. The numerical solution of parabolic and elliptic differential equations. J. Soc. Indust. Appl. Math. 3 (1955), 28–41.

The authors attack the partial differential equation

$$U_t=U_{xx}+U_{yy} \tag{1}$$

by difference methods. Cover the region of integration with a mesh $x=i\Delta x, y=j\Delta y, t=n\Delta t$ $(\Delta x=\Delta y, \varrho=(\Delta x)^2/\Delta t)$; let $U_n=U(i\Delta x, j\Delta y, n\Delta t)$ and let $\Delta_x^2U_n$, $\Delta_y^2U_n$ be divided second central differences at time stop n. Compute U_{n+1}, U_{n+2} from the formulas

$$\frac{U_{n+1}-U_n}{\Delta t}=\Delta_x^2U_{n+1}+\Delta_y^2U_n, \tag{2}$$

$$\frac{U_{n+2}-U_{n+1}}{\Delta t}=\Delta_x^2U_{n+1}+\Delta_y^2U_{n+2}. \tag{3}$$

U_{n+1}, U_{n+2} are readily obtained by factoring the corresponding coefficient matrices in (2), (3). The process is stable (round-off errors do not grow) for all mesh-ratios ϱ, so that only truncation error need be considered in choosing $\Delta x=\Delta y$ and Δt. For time-independent boundary conditions, U_n approaches the solution of Laplace's difference equation $(\Delta_x^2+\Delta_y^2)U=0$.

The authors discuss in detail solution over a square, and show a considerable saving in work for their method over the usual ones. For elliptic equations, their method is a form of "line relaxation" and appears more efficient than any of the usual "point-relaxation" schemes. Independently M. E. Rose and H. Keller have developed a line-relaxation technique, to be published later.
M. A. Hyman (Philadelphia, Pa.).

Douglas, Jim, Jr. On the numerical integration of

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=\frac{\partial u}{\partial t}$$

by implicit methods. J. Soc. Indust. Appl. Math. 3 (1955), 42–65.

The author considers solution of $u_t=u_{xx}+u_{yy}$ with rather general initial and boundary conditions. Using mostly Fourier methods, he discusses convergence of three implicit difference schemes; for notation, see the preceding review. The first scheme is that of Peaceman and Rachford described above. The second and third schemes are extensions to two space dimensions of schemes given by O'Brien, Hyman, and Kaplan [J. Math. Phys. 23 (1951) 223–251; MR **12**, 751]:

$$\frac{u_{n+1}-u_n}{\Delta t}=(\Delta_x^2+\Delta_y^2)u_{n+1}$$

and

$$\frac{u_{n+1}-u_n}{\Delta t}=\tfrac{1}{2}(\Delta_x^2+\Delta_y^2)(u_n+u_{n+1}).$$

M. A. Hyman (Philadelphia, Pa.).

Philip, J. R. Numerical solution of equations of the diffusion type with diffusivity concentration-dependent. Trans. Faraday Soc. 51 (1955), 885–892.

The author wishes to solve, for $x>0$, $t>0$:

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial\theta}{\partial x}\right),\quad \theta(x,0)=\theta_n,\ \theta(0,t)=\theta_0,\ \theta_0>\theta_n, \tag{1}$$

where D is a single-valued function of θ. Introducing the variable $\phi=xt^{-\frac{1}{2}}$, one is led to the relation

$$d\phi=-2D\,d\theta\Big/\int_{\theta_n}^{\theta}\phi\,d\theta, \tag{2}$$

where θ, ϕ are, respectively, the independent and dependent variables. Auxiliary conditions on $\phi(\theta)$ are

$$\phi(\theta_0)=0,\ \phi(\theta_n)=\infty.$$

The interval $\theta_0 \geqq \theta \geqq \theta_n$ is split into n parts, in each of which $D(\theta)$ is replaced by its average value. A simple iterative technique is described for computing $\phi(\theta_r)$ ($r=1, 2, \cdots, n-1$). For $r=n-1$, the numerical solution must match the analytic solution of (2) in the interval $\theta_{n-1} \geqq \theta \geqq \theta_n$; the latter involves a new function $A(x)$, formed from the error function and its integral. The asymptotic expansion of $A(x)$ is given, plus a short table. The calculation is repeated, halving $\Delta\theta$, and the truncation error reduced using Richardson's h^2-process. Several numerical examples support the author's claim that the method is usually rapidly convergent and yields an accurate solution of problem (1). *M. A. Hyman.*

Džems-Levi, G. E. The projective transformation of nomograms. Uspehi Mat. Nauk (N.S.) 7 (1952), no. 4(50), 147–151. (Russian)

This is an expository article on Pentkovskiĭ's [Nomografiya, Gostehizdat, Moscow-Leningrad, 1949; MR 13, 78] method of transforming nomograms by means of networks with three fixed points. *E. Lukacs.*

Gary, Max. Die Konstruktion der mittleren Linie bei Oberflächenprofilen und einige Zusammenhänge zwischen den zur Kennzeichnung von Oberflächen benutzten Grössen. Z. Angew. Math. Physik 5 (1954), 490–496.

In order to obtain information for a given surface the author investigates intersections by planes perpendicular to a given plane. A graphical procedure using a planimeter for obtaining the "mean line" to a segment of such an intersection is given. This line is discussed in relation to the Fourier series of the segment. *F. J. Murray.*

See also: Bartle, p. 176; Ščigolev and Reznikovskiĭ, p. 198.

Tables

Schmidt, Paul W. Tables of $\int_0^x J_0(t)dt$ for large x. J. Math. Phys. 34 (1955), 169–172.

Table of $\int_0^x J_0(t)dt$ for $x=10(0.2)40$ (6D), with second differences. For $10 \leqq x \leqq 20$ the table was constructed by numerical integration, and for $20 < x \leqq 40$ an asymptotic expansion was used. *J. A. Todd.*

Salzer, Herbert E. Complex zeros of the error function. J. Franklin Inst. 260 (1955), 209–211.

The error function $f(z)=\int_0^z e^{-u^2}du$ has no real zeros, other than the trivial $z=0$, but has an infinite number of complex zeros symmetrically situated in the four quadrants of the z-plane. In the first quadrant they all lie above the line $x=y$ and approach this line asymptotically. The main purpose of this note is the tabulation of the first ten zeros z_n of $f(z)$. The first zero is given to twelve decimals and the others to nine. There is a short discussion on the use of the table for calculating the inverse error function in the neighbourhood of the zeros. *L. Fox.*

See also: Astuni, p. 120; Masuyama, p. 169.

Mathematical Machines

DiPrima, Richard C. Solution of some eigenvalue problems on the EDVAC. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 924 (1955), 26 pp.

Es handelt sich um eine Methode zur Berechnung von Eigenwerten unter Verwendung von Grossrechenmaschinen. Der Verfasser schlägt vor, die nach der Berechnung der partikulären Rechnungen sich ergebende Gleichung für den Eigenwert, nach einem Verfahren aufzulösen, das der Newton'schen Näherungsmethode nachgebildet ist, nur dass dabei der Differentialquotient durch einen Differenzenquotienten ersetzt wird. Die für die Berechnung des Differenzenquotienten benötigten ersten beiden Näherungswerte müssen auf anderem Wege ermittelt werden. Erläutert wird das Verfahren an drei Beispielen. Das erste betrifft die Theorie des schwingenden Stabes, das zweite die Theorie der Gleitlager und das dritte ein Problem, das sich beim Studium der Stabilität einer zähen Flüssigkeit zwischen zwei rotierenden unendlich langen konzentrischen Zylindern ergibt und auf eine nicht selbstadjungierte Differentialgleichung 6. Grades führt. Das zweite Beispiel ist deshalb von numerischem Interesse, weil zwei Eigenwerte sehr nahe beisammen liegen. *P. Funk* (Wien).

Davis, Chandler. Linear programming and computers. I, II. Computers and Automation 4 (1955), no. 7, 10–17; no. 8, 10–16.

Booth, Andrew D. Use of a computing machine as a mechanical dictionary. Nature 176 (1955), 565.

Eckert, W. J. The significance of the new computer NORC. I. Computers and Automation 4 (1955), no. 2, 10–13.

Wilkinson, J. H. An assessment of the system of optimum coding used on the pilot automatic computing engine at the National Physical Laboratory. Philos. Trans. Roy. Soc. London. Ser. A. 248 (1955), 253–281.

Hopper, G. M. Automatic coding for digital computers. Computers and Automation 4 (1955), no. 9, 21–24.

Hume, J. N. P., and Worsley, Beatrice H. Transcode: a system of automatic coding for FERUT. J. Assoc. Comput. Mach. 2 (1955), 243–252.

Gorman, T. P., Kelly, R. G., and Reddy, R. B. Automatic coding for the IBM 701. J. Assoc. Comput. Mach. 2 (1955), 253–261.

Forbes, George F. The digital differential analyzer. Computers and Automation 4 (1955), no. 1, 8–10.

Worsley, Beatrice H. Numerical representation in fixed-point computers. Computers and Automation 4 (1955), no. 5, 10–13.

Atkinson, Cyril P. A mechanical analog computer for solving linear simultaneous algebraic equations. Computers and Automation 4 (1955), no. 3, 12–15, 30.

Ugodčikov, A. G. Electromodelling of the problem of conformal mapping of a circle on a simply connected region given beforehand. Ukrain. Mat. Ž. 7 (1955), 221–230. (Russian)

Trapeznikov, V. A. Mathematical modelling of dynamical systems. Èlektričestvo no. 8 (1955), 20–26. (Russian)

Kapica, S. P. An apparatus for computation of the Poisson integral and some of its applications. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 369–376. (Russian)

Haneman, V. S., and Senders, J. W. Correlation computation on analog devices. J. Assoc. Comput. Mach. 2 (1955), 267–279.

Braier, A. Un appareil électrique pour l'analyse harmonique. Gaz. Mat. Fiz. Ser. A. 7 (1955), 322–327. (Romanian. Russian and French summaries)

Marner, Gene R. High precision computer for automatic solution of the celestial triangle. Navigation 4 (1955), 281–285.

Bonfiglioli, L. An instrument for finding the line of intersection of two curved surfaces. Technion. Israel Inst. Tech. Sci. Publ. 6 (1954/5), 14–33. (Hebrew summary)

Pajares Díaz, Emilio. A prism tangent drawer. Rev. Ci. Apl. 9 (1955), 122–129. (Spanish)

See also: Booth, p. 126.

ASTRONOMY

Mendes, Marcel. Nouvelle démonstration des équations de Lagrange. Bull. Astr. 18 (1954), 69–72. (English, German and Russian summaries)

Use is made of a theorem of Chazy, generalizing the usual theorem on the invariance of Hamiltonian equations under contact transformations, to derive the equations for the variations of the elliptic elements of a disturbed planet. *D. C. Lewis* (Baltimore, Md.).

Moiseev, N. D. On interpolation-averaged variants of the canonical problem of celestial mechanics. Moskov. Gos. Univ. Trudy Gos. Astr. Inst. 24 (1954), 3–9. (Russian)

In the three-dimensional case the Hamiltonian is averaged for some interval over a plane $\alpha=\sum_{i=1}^{3} a_i g_i$ determined from observations by least squares, say, and then this mean Hamiltonian used in the canonical equations of motion. The canonical coordinates g_i now enter only through the linear form for α, making the differential equations somewhat simpler, and two independent integrals are immediately obtained. This same averaging procedure for obtaining an approximate orbit is also worked out by the author using a fitted line instead of a plane. *R. G. Langebartel* (Urbana, Ill.).

Moiseev, N. D. On complete averaging of the canonical problem of celestial mechanics with several intermediary elements. Moskov. Gos. Univ. Trudy Gos. Astr. Inst. 24 (1954), 10–16. (Russian)

The procedures detailed in the paper reviewed above are extended to the n-dimensional case.
R. G. Langebartel (Urbana, Ill.).

Sklyanskiĭ, A. L. On trajectories of a double collision in the problem of three bodies interacting with forces proportional to the logarithms of the mutual distances. Ukrain. Mat. Ž. 6 (1954), 349–362. (Russian)

Yu. D. Sokolov [same Ž. 2 (1950), no. 4, 25–36; MR 13, 996] discussed the rectilinear motion of three particles, mutually attracting or repulsing under the action of forces with moduli

$$(1)\qquad g^2 m_i m_j |\ln r_k/a| \quad (i, j, k=0, 1, 2; i\neq j\neq k),$$

where r_k denotes the distance between the particles P_i and P_j, g^2 and a being two positive constants. In the present paper the results obtained by Sokolov concerning a double collision between a definite pair of the particles are generalized to the motion of three particles in space, assuming that the interactions are subject to the same law (1). *E. Leimanis* (Vancouver, B.C.).

Agostinelli, Cataldo. Sull'esistenza di soluzioni periodiche nel problema ristretto dei tre corpi. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88** (1954), 265–291.

It is shown that, under the usual condition of nonresonance, the restricted problem of three bodies in the plane admits a double infinitude of essentially distinct periodic motions for each sufficiently small value of the parameter μ. The method involves detailed explicit calculation of the lower terms in the series expansion of the general solution in powers of μ and the initial values $\beta_1, \beta_2, \beta_3, \beta_4$ of the variations from the generating circular Keplerian motion. An expansion of the perturbation function in Legendre polynomials is utilized in this calculation. The origin of time is specified by the condition $\beta_4=0$, and then the arbitrary parameters of the family are β_2 and the period T of the generating circular solution.
D. C. Lewis (Pacific Palisades, Calif.).

Ščigolev, B. M. Intermediary orbits in the problem of three bodies. Moskov. Gos. Univ. Trudy Gos. Astr. Inst. 24 (1954), 59–92. (Russian)

A survey of the various types of approximate orbits for the three-body problem proposed by D'Alembert, Poisson, Gyldén, Hill, Clairaut, Laplace, Schmeidler, Moiseev, and others. *R. G. Langebartel* (Urbana, Ill.).

Ščigolev, B. M., and Reznikovskiĭ, P. T. On the application of Runge's method to the numerical solution of the equations of celestial mechanics. Moskov. Gos. Univ. Soobšč. Astr. Inst. no. 92 (1953), 3–22. (Russian)

The authors outline the method of Runge for numerical integration of differential equations and work out a number of variants especially useful for starting a table for the types of equations usually met with in celestial mechanics. *R. G. Langebartel* (Urbana, Ill.).

Colleau, Jean. Sur la stabilité d'une planète constituée d'une matière présentant une discontinuité de densité. Bull. Astr. 18 (1954), 193–232. (English, German and Russian summaries)

The stability under a small radial deformation of an incompressible fluid sphere with a spherical core of density $1+kd$ surrounded by a fluid with density d is studied. The model is found to be unstable for $k>\frac{1}{2}$.
R. G. Langebartel (Urbana, Ill.).

Cicin, F. A. A simple proof of V. G. Fesenkov's theorem. Astr. Ž. 31 (1954), 80. (Russian)

In an oblate spheroidal mass distribution in which the surfaces $r^2/(ka)^2+z^2/(kc)^2=1$ are constant-density surfaces

for each k, $0\leq k\leq 1$, the ratio of the mass in a unit solid angle from a point in the equatorial plane in the direction in the equatorial plane perpendicular to the diameter through the point to the mass in the direction from this point perpendicular to the equatorial plane is $(a/c)^3$, where a is the semi-major axis and c the semi-minor axis. The present proof rests on changing the z scale to transform the ellipsoid into a sphere. *R. G. Langebartel.*

Roy, T. C. Consideration of energy flow in the vibration of a stellar model. Bull. Calcutta Math. Soc. **46** (1954), 87–92.

In this first-order theory the author has employed an energy equation in place of the usual adiabatic assumption. To facilitate the mathematical analysis the opacity is assumed proportional to $\varrho^{-5/2}T^3$ where ϱ is the density and T is the temperature. In this case a stationary vibration gives rise to the presence of a node in the velocity vibration at the surface of the star. *R. G. Langebartel.*

Gurevič, L. È. Bilateral interaction of stars with a diffusing substance and its significance in a cosmogonical process. Voprosy Kosmog. 3 (1954), 94–224. (Russian)

The absorption and radiation of a star inside a diffuse nebula is given a qualitative study. In particular, there is presented a theory of novae in such nebulae. It is proposed that a star traveling sufficiently slowly through a diffuse nebula containing deuterium to an atomic concentration of about 10^{-4} will absorb from the nebula matter which will, however, stay at the star's surface. This deuterium-containing matter, heated by the radiation from the star's interior, will under certain conditions undergo a nova explosion. The star must absorb a mass proportional to the cube of its radius and inversely proportional to the square of the surface temperature for this to occur; thus hot dense stars are favored. The author concludes that the probability of such an explosion for the sun or a star similarly located in the spectrum-luminosity diagram is negligible, contrary to the prediction of Hoyle's theory. *R. G. Langebartel.*

Sobolev, V. V. The theory of scattering of light in planetary atmospheres. Uspehi Astr. Nauk **6** (1954), 250–280. (Russian)

A survey of recent research by Russian authors on the theory of multiple scattering of light in extended regions, principally the works of Ambartsumian, Kuznetsov, and Sobolev. *R. G. Langebartel* (Urbana, Ill.).

Plumpton, C., and Ferraro, V. C. A. On toroidal magnetic fields in the sun and stars. Astrophys. J. **121** (1955), 168–174.

A first-order theory governing the oscillations of a spherical fluid with infinite electrical conductivity in a permanent axially symmetric magnetic field is evolved. Axially symmetric oscillations (with a continuous frequency spectrum) whose cross sections are successive lines of force are found to occur. *R. G. Langebartel.*

Khare, R. C. The expansion of a gas-cloud into a vacuum. Z. Astrophys. 33 (1954), 251–266.
Khare, R. C. The expansion of a semi-infinite gas-cloud into a vacuum. Z. Astrophys. **35** (1954), 115–124.
Khare, R. C. The expansion of an interstellar gas-cloud, viscosity taken into account. Z. Astrophys. **35** (1954), 125–134.

These three papers deal with the motion of a plane stratified gas under isentropic conditions. In the first paper, the author reduces the problem to the usual partial differential equation

$$\frac{\partial^2 w}{\partial r\partial s}+\frac{n}{r+s}\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right)=0,$$

where r, s are the Riemann variables. He rediscovers various forms for the solution of this equation, in particular, those involving hypergeometric functions. He applies a solution of the latter type to the specific case of the expansion of a semi-infinite gas cloud into a vacuum, there being a non-homogeneous region at the start of the motion at the outer edge of the gas.

In the second paper the author uses a new method for the solution of the basic equation for the flow. He introduces a velocity potential, φ, finds the equation satisfied by φ and solves this by breaking it up into two parts that he equates separately to zero. He does not use the most general solution of the resulting pair of equations, but only one in which the velocity is a linear function of x/t, where x is the spatial coordinate and t, apart from an additive constant, is the time. This solution is again applied to the expansion into a vacuum of a semi-infinite gas cloud with a heterogeneous layer at the outer face. In the third paper, Khare proposes a solution for the flow of a viscous gas, by the method he has employed in the second paper. However, this method implies that the velocity, u, is a linear function of x whereas viscosity enters into the equation of motion by means of a term $\alpha\,\partial^2u/\partial x^2$. This term is therefore identically zero for all solutions, and not only for that one in which the ratio of specific heats has the value 3, as stated by Khare. The specific case worked out is again that of the expansion of a gas-cloud into a vacuum. *G. C. McVittie.*

See also: Gilbert, p. 202; Simon, p. 208.

RELATIVITY

Einstein, A., and Kaufman, B. A new form of the general relativistic field equations. Ann. of Math. (2) **62** (1955), 128–138.

The authors give two derivations of the field equations of Einstein's generalized theory of relativity. In both of these a variational principle is used. In the first method an auxiliary vector field is introduced and then eliminated. In the second method a "pseudo-tensor" field related to the second nonsymmetrical affine connection is used. In the formulation various identities and conservation laws are deduced from the invariance of the action integral under different types of transformations. *A. H. Taub.*

★ **Lichnerowicz, A. Théories relativistes de la gravitation et de l'électromagnétisme. Relativité générale et théories unitaires.** Masson et Cie, Paris, 1955. xii+298 pp.

In this book the author has given a rigorous mathematical treatment of a number of specialized problems in the general theory of relativity. The point of view, throughout, is that of classical continuum mechanics. The physical system is characterized by gravitational and electromagnetic fields without singularities and by an energy-momentum tensor, the material energy-momentum tensor usually describing incoherent matter or a

perfect fluid. The most striking feature of the book is the careful study of the Cauchy initial-value problem for many of the physical systems considered.

The book is divided into five sections. The first is on the "Principles of general relativity", the second is on relativistic hydrodynamics and is entitled "Rotational and irrotational motions," the third is on "Global studies of stationary space-times." The last two sections are on unified field theories: "The theory of Jordan-Thiry" deals with generalizations of Kaluza's 5-dimensional relativity, "The theory of Einstein-Schrödinger" deals with the recent theories of these authors.

It should perhaps be pointed out that the book is not suitable for the mathematician or physicist who is unfamiliar with general relativity and who would like to have an idea of what the subject is about. Lichnerowicz does not mention the equivalence principle or any of the other physical ideas which led Einstein to the formulation of his theory of gravitation. There is no discussion of the Schwarzschild metric, of planetary motion, or of the important fact that Newton's theory of gravitation is a good first approximation to Einstein's. *A. Schild.*

Thiry, Yves. Sur le caractère statique d'un modèle d'univers stationnaire en théorie unitaire de Jordan-Thiry. C. R. Acad. Sci. Paris **241** (1955), 691–692.

In general relativity, space-times which have stationary metrics (metrics independent of the time) are asymptotically Euclidean, and those which are such that the particle paths coincide with the curves of parameter t, the time, are such that these curves are orthogonal to the surfaces t=constant [see the book reviewed above, p. 136]. In this paper the author outlines a proof of the corresponding theorem for the unified field theory of Jordan-Thiry. *A. H. Taub* (Urbana, Ill.).

Just, Kurt. Neue Feldgleichungen zur Jordanschen Gravitations-Theorie. Z. Physik **140** (1955), 485–493.

A class of variational principles which includes that from which the Jordan theory of graviation may be derived is studied. The field equations corresponding to various choices of certain parameters are enumerated. *A. H. Taub* (Urbana, Ill.).

Just, Kurt. Zur Planetenbewegung bei veränderlicher Gravitationszahl. Z. Physik **140** (1955), 524–530.

The spherically symmetric, static solution of the class of field equations discussed in the paper reviewed above is discussed. It is shown that all of these can in first approximation lead to the Einstein formula for the advance of the perihelion of a planet. *A. H. Taub* (Urbana, Ill.).

Nariai, Hidekazu. Some remarks on Jordan's projective relativity. Sci. Rep. Tôhoku Univ. Ser. 1. **37** (1953), 423–430 (1954).

The author discusses particular solutions of the field equations derived from the variational principle

$$\int K^{\eta}(G-\xi K^{-2}K_{\mu}K^{|\mu})\sqrt{(-g)}dx^1dx^2dx^3dx^4=0,$$

where g_{ij} is the metric tensor of space-time, G is the scalar curvature formed from this tensor, K is a scalar field to be interpreted as the "gravitational constant," $K_{|\mu}=\partial K/\partial x^{\mu}$, $K^{|\mu}=g^{\mu\nu}K_{|\nu}$, and η and ξ are constants. Jordan's field equations are obtained by setting $\eta=1$. The particular solutions discussed are of two types:

(a) $$ds^2=-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+e^{\nu}dt^2$$

and K depending only on r, with λ and ν functions of r alone, and

(b) $$ds^2=c^2dt^2-\frac{\delta R^2(t)}{(1+\frac{1}{4}er^2)}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2).$$

The solutions of the type (a) are shown to be Heckmann's solution of Jordan's field equations [Heckmann, Jordan, and Fricke, Z. Astrophys. **28** (1951), 113–149; MR **15**, 656]. Various possible cosmological solutions of type (b) are discussed. *A. H. Taub* (Urbana, Ill.).

Hély, Jean. Sur la représentation du champ unitaire par un tenseur g_{ik} non symétrique. C. R. Acad. Sci. Paris **241** (1955), 645–647.

Let $g_{\lambda\mu}$ ($f_{\lambda\mu}$) be a general (symmetric) tensor with the negative determinant $\mathfrak{g}(\mathfrak{f})$ and $*g^{\lambda\nu}(*f^{\lambda\nu})$ the inverse tensor to $g_{\lambda\mu}$ ($f_{\lambda\mu}$). Denote by $F_{\lambda\mu}$ the contracted curvature tensor of $f_{\lambda\mu}$ and put

$$G_{\lambda\mu}\overset{\text{def}}{=}F_{\lambda\mu}-\tfrac{1}{2}F_{\alpha\beta}*f^{\alpha\beta}f_{\lambda\mu},\quad k_{\lambda\mu}\overset{\text{def}}{=}g_{[\lambda\mu]},\quad *k^{\lambda\nu}\overset{\text{def}}{=}*g^{[\lambda\nu]},$$
$$h_{\lambda\mu}\overset{\text{def}}{=}g_{(\lambda\mu)},\quad *h^{\lambda\nu}\overset{\text{def}}{=}*g^{(\lambda\nu)}.$$

If

(I) $$(\mathfrak{g}/\mathfrak{f})^{\frac{1}{2}}\,*h^{\lambda\nu}=*f^{\lambda\nu},\quad h_{\mu\nu}=\varepsilon f_{\mu\nu}+\lambda^{-1}F_{\mu\nu}$$

or

(II) $$h_{\lambda\nu}=f_{\lambda\nu},\quad (\mathfrak{g}/\mathfrak{f})^{\frac{1}{2}}\,*h^{\nu\mu}=\varepsilon*f^{\nu\mu}+\lambda^{-1}G_{\alpha\beta}\,*f^{\alpha\nu}\,*f^{\beta\mu}$$
$$(\varepsilon,\ \lambda=\text{const.}),$$

then a simple inspection shows that

$$2k_{\lambda\mu}((-\mathfrak{g})^{\frac{1}{2}}\,*k^{\mu\nu})_{,\nu}+3(-\mathfrak{g})^{\frac{1}{2}}\,*k^{\mu\nu}k_{[\mu\nu,\lambda]}=0.$$

Hence if

(III) $$k_{[\mu\nu,\lambda]}=0,$$

then

$$k_{\lambda\mu}((-\mathfrak{g})^{\frac{1}{2}}\,*k^{\mu\nu})_{,\nu}=0$$

and if

(IV) $$((-\mathfrak{g})^{\frac{1}{2}}\,*k^{\mu\nu})_{,\nu}=0,$$

then

$$*k^{\mu\nu}k_{[\mu\nu,\lambda]}\neq 0.$$

In the case III (IV) the tensors $k_{\lambda\mu}$ and $*k^{\lambda\nu}(\mathfrak{g}/\mathfrak{f})^{\frac{1}{2}}$ may be obviously identified with the Born tensors $B_{\lambda\mu}$, $H^{\lambda\nu}$ (with the dual to Born tensors) which characterize the electromagnetic field.

Remark of the reviewer. Let $\mathfrak{h}<0$ be the determinant of $h_{\lambda\mu}$, which is here the metric tensor. The tensor $*g^{\lambda\nu}$ may be expressed in terms of $g_{\lambda\mu}$. Thus, in particular,

$$*h^{\lambda\nu}=\frac{\mathfrak{h}}{\mathfrak{g}}[h^{\lambda\nu}(1+\tfrac{1}{2}k_{\alpha\beta}k^{\alpha\beta})+k^{\lambda\alpha}k_{\alpha}{}^{\nu}]$$

[cf. Hlavatý, Nieuw Arch. Wisk. (3) **2** (1954), 103–114; especially p. 105, where $k^{\nu}{}_{\alpha}$ should read $k_{\alpha}{}^{\nu}$; MR **16**, 408]. Therefore I (or II) imposes strong conditions on the tensor $g_{\lambda\mu}$. Thus for instance II is equivalent to

$$\frac{1}{\lambda}G^{\nu\mu}=h^{\nu\mu}\Big[\Big(\frac{\mathfrak{h}}{\mathfrak{g}}\Big)^{\frac{1}{2}}(1+\tfrac{1}{2}k_{\alpha\beta}k^{\alpha\beta})-\varepsilon\Big]+\Big(\frac{\mathfrak{h}}{\mathfrak{g}}\Big)^{\frac{1}{2}}k^{\lambda\alpha}k_{\alpha}{}^{\nu}$$

while the condition I gives rise to a still more complicated condition for $g_{\lambda\mu}$. *V. Hlavatý* (Bloomington, Ind.).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. B_3. J. Rational Mech. Anal. **4** (1955), 653–679.

Of the author's series on the unified theory, this paper is a direct continuation of paper B_2 [same J. **4** (1955),

247–277; MR **16**, 870]. The problem here discussed is the algebraic one of finding the solution $\Gamma^{\nu}_{\lambda\mu}$ of

$$D_{\omega}g_{\lambda\mu}=2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha},$$

where D_{ω} denotes covariant differentiation with respect to the Γ's and $S_{\omega\mu}{}^{\alpha}$ the torsion tensor, in two exceptional cases. These are the cases in which the ratio g of the determinants $|g_{\lambda\mu}|$ and $|g_{(\lambda\mu)}|$ is equal either to zero or to 2. As a basis for his attack on the problem, the author investigates the properties of the Nijenhuis tensor defined by

$$\tfrac{1}{2}N_{\lambda\mu}{}^{\nu}=(\partial_{\alpha}k_{[\lambda}{}^{\nu})k_{\mu]}{}^{\alpha}-k_{\alpha}{}^{\nu}\partial_{[\mu}k_{\lambda]}{}^{\alpha},$$

$k_{\lambda\mu}$ being the skew-symmetric part of $g_{\lambda\mu}$ [Nijenhuis, Nederl. Akad. Wetensch. Proc. Ser. A. **54**=Indag. Math. **13** (1951), 200–212; MR **13**, 281]. In particular, he obtains the relationship between $N_{\lambda\mu}{}^{\nu}$ and $S_{\lambda\mu}{}^{\nu}$. It turns out, incidentally, that the condition $S_{\lambda\mu}{}^{\nu}=0$ reduces space-time locally to a Riemannian product-space in which every pair of eigenvectors of $k_{\lambda}{}^{\nu}$ is X_2-forming. In solving his main problem, the author obtains results of which the following is typical: if $g=0$, a necessary and sufficient condition for the existence of a solution is that the eigenvectors ${}_3e^{\lambda}$, ${}_4e^{\lambda}$ be X_2-forming. *H. S. Ruse* (Leeds).

Treder, Hans-Jürgen. Der Materietensor in der unsymmetrischen Feldtheorie Einsteins. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **4** (1955), 9–10.

[In this review we use the abbreviations B, C_2, C_3, B_2, B_3, U, I, M for the following papers of the reviewer: "The elementary basic principles of the unified theory of relativity", B, C_2, C_3, B_2, B_3, J. Rational Mech. Anal. **2** (1953), 1–52; **3** (1954) 103–146, 147–149, 647–689; **4** (1955), 247–277, 653–674 [MR **14**, 505; **15**, 654; **16**, 408, 870; and the preceding review] "Uniqueness theorems in the unified theory of relativity" (with A. W. Sáenz), ibid. **2** (1953), 523–536 [MR **14**, 1132] "The law of inertia in the unified field theory". Univ. e Politec. Torino. Rend. Sem. Mat. 13 (1954), 153–167 [MR **16**, 755]; "Maxwell's field in the Einstein unified field theory", Nieuw Arch. Wisk. (3) 2 (1954), 103–114 [MR **16**, 408].]

If the Einstein system

$$\partial_{\omega}g_{\lambda\mu}-\Gamma_{\lambda\omega}{}^{\alpha}g_{\alpha\mu}-\Gamma_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}=0$$

admits a unique solution (for necessary and sufficient conditions see U, p. 528), then this solution has the form

$$\Gamma_{\lambda\mu}{}^{\nu}=\left\{ {\nu \atop \lambda\,\mu} \right\}+S_{\lambda\mu}{}^{\nu}+U^{\nu}{}_{\lambda\mu}. \tag{1}$$

Here $\left\{ {\nu \atop \lambda\,\mu} \right\}$ are the Christoffel symbols of $h_{\lambda\mu}\overset{\text{def}}{=}g_{(\lambda\mu)}$ and the tensors $S_{\lambda\mu}{}^{\nu}$ and $U^{\nu}{}_{\lambda\mu}$ are well defined explicit functions of $h_{\lambda\mu}$ and $k_{\lambda\mu}\overset{\text{def}}{=}g_{[\lambda\mu]}$ and their first derivatives (see B, pp. 5 and 13). Substituting into the Einstein condition $R_{(\lambda\mu)}=0$ for the contracted curvature tensor $R_{\lambda\mu}$ of $\Gamma_{\lambda\mu}{}^{\nu}$ from (1), one obtains an equivalent condition

$$H_{\lambda\mu}-\tfrac{1}{2}Hh_{\lambda\mu}=T_{\lambda\mu} \tag{2}$$

(see C_2, p. 155, 169; C_3, p. 663, 673 in a slightly different notation). Here $H_{\lambda\mu}$ is the contracted curvature tensor of $h_{\lambda\mu}$, H the corresponding scalar curvature and $T_{\lambda\mu}$ (the total momentum-energy tensor) is an explicitly defined function of $h_{\lambda\mu}$, the Maxwell field $m_{\lambda\mu}$ (see M) and their first derivatives. Therefore the conservation law holds

$$\nabla_{\lambda}T^{\lambda\nu}=0, \tag{3}$$

where ∇_{λ} is the symbol of the covariant derivative with respect to $\left\{ {\nu \atop \lambda\,\mu} \right\}$. This equation leads to the equations of motion (see C_2, p. 171; C_3 p. 676; I, p. 5).

The author, unaware of these results, rediscovers them (formally), chiefly by means of a variational principle and shows that (3) is equivalent to

$$\partial_{\lambda}(\mathfrak{T}_{\mu}{}^{\lambda}+\mathfrak{t}_{\mu}{}^{\lambda})$$

where $\mathfrak{T}_{\mu}{}^{\lambda}=T_{\mu}{}^{\lambda}|\mathfrak{h}|^{\frac{1}{2}}$ and $\mathfrak{t}_{\mu}{}^{\lambda}$ is built in the usual way from the Lagrange function

$$|\mathfrak{h}|^{\frac{1}{2}}h^{\alpha\beta}\left[\left\{ {\gamma \atop \delta\,\beta} \right\}\left\{ {\delta \atop \alpha\,\gamma} \right\}-\left\{ {\gamma \atop \delta\,\gamma} \right\}\left\{ {\delta \atop \alpha\,\beta} \right\}\right].$$

[Remark of the reviewer: The author's statement that $S_{\lambda\mu}{}^{\nu}+U^{\nu}{}_{\lambda\mu}$ cannot be expressed explicitly in terms of $h_{\lambda\mu}$, $k_{\lambda\mu}$ and their first derivatives is not correct. Cf. B_2, pp. 261, 266, 275 and 276; for the degenerate cases cf. B_3, Part II.] *V. Hlavatý* (Bloomington, Ind.).

Tonnelat, Marie-Antoinette. Sur les équations approchées de la théorie d'Einstein-Schrödinger. C. R. Acad. Sci. Paris, **241** (1955), 168–170.

The author evaluates the field equations of Einstein's generalized theory of relativity to second-order terms. They then become second-order partial differential equations for the symmetric part (gravitational) anti-symmetric part (electromagnetic) of the fundamental non-symmetric tensor $g_{\mu\nu}$ which underlies this theory. *A. H. Taub* (Urbana, Ill.).

Infeld, L. Equations of motion for linear field-theories. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 213–216.

The author reviews the derivation of the equations of motion of a single singularity in a gravitational field from the field equations. When another field in addition to the gravitational one is present, an analogous form for the equations of motion exists in general relativity but not in special relativity. Even in the latter case he proposes to assume as equations of motion of a singularity in a linear field theory equations of the form suggested by the general theory of relativity. *A. H. Taub* (Urbana, Ill.).

Pham Tan Hoang. L'emploi de la métrique $h^{\mu\nu}$ et des champs $f^{\mu\nu}$ pour l'obtention des équations du mouvement. C. R. Acad. Sci. Paris **241** (1955), 170–172.

The author shows that the Einstein-Hoffman-Infeld approximation scheme when applied to the Einstein generalized theory of relativity, where the gravitational field is associated with the symmetric part of the contravariant fundamental non-symmetric tensor $g^{\mu\nu}$ and the electromagnetic field is associated with the antisymmetric part of this same tensor, leads to equations of motion of singularities of the field which are independent of the electromagnetic field strengths. Callaway [Phys. Rev. (2) **92** (1953), 1567–1570; MR **15**, 564] had previously obtained a similar result in terms of the association of the gravitational and electromagnetic fields with the covariant form of the fundamental tensor. *A. H. Taub.*

Bertotti, B. On the two-body problem in general relativity. Nuovo Cimento (9) **12** (1954), 226–232.

The problem is similar to that investigated by Eddington and Clark [Proc. Roy. Soc. London. Ser. A. **166** (1938), 465–475], that is of finding the equations of motion of a double star from the geodetic principle; the deduction is simple and the known equations of motion are obtained as a result. *L. Infeld* (Warsaw).

Gilbert. C. The stability of a spherically symmetric cluster of stars describing circular orbits. Monthly Not. Roy. Astr. Soc. **114** (1954), 628–634 (1955).

A spherical system of gravitating particles in a steady state, each particle moving in a circular orbit about the centre of symmetry, is discussed using general relativity. The energy-tensor for the system is found and the equations for the coefficients of the metric established. The internal solution that is obtained is fitted to the external Schwarzschild solution appropriate to empty space. The condition of stability for the orbits is found and it is proved that the velocity of a particle at the outer boundary of the system must be less than half the velocity of light. The proper-mass of the system is calculated and compared with its gravitational mass (i.e. with the mass appearing in the external Schwarzschild solution). The gravitational mass of the system is, in most cases of interest, less than its proper-mass when the orbits are stable. These systems are of two kinds: in one the particle density is not zero at the outer boundary and the particle-velocities increase with distance from the centre; in the other, the particle-density is zero at the boundary and the highest velocites occur near the centre. Both kinds are stable. The second kind seems to provide the better analogy with globular clusters of stars. In formula (3), g_{ik} is misprinted; in formula (8), the right-hand side should be $(e^{\lambda}-1)/r$. *G. C. McVittie* (Urbana, Ill.).

Kalicin, N. S. Relativistic mechanics of a material point of variable mass. Ž. Eksper. Teoret. Fiz. **28** (1955), 631–632. (Russian)

The equations of motion in the special relativity theory are derived for a particle the mass of which varies because of the continuous addition to it of mass having a given velocity. *N. Rosen* (Haifa).

Littlewood, D. E. The cosmological principle. Proc. Cambridge Philos. Soc. **51** (1955), 678–683.

The perfect cosmological principle is analysed and shown to lead to the metric of the de Sitter universe. Attention is paid to the assumption that the distances between galaxies remain constant if a certain time-scale is used. It is not explained if this distance is that defined by astronomical measurements or is some purely mathematical distance-parameter. The working-out is based on the introduction of a function θ for which no interpretation is given and whose connection with the postulate of constant distances is not clear to this reviewer.
G. C. McVittie (Urbana, Ill.).

Schlüter, Arnulf. The cosmical constant and the classical limit of Einstein's field equations. Astr. J. **60** (1955), 141–143.

By considering the case of a metric with spherical symmetry it is proved that the cosmical constant can, under certain conditions, survive when Einstein's equations degenerate to the Newtonian case. The effect of the cosmical constant is then equivalent to that of a universal substratum of matter at constant density.
G. C. McVittie (Urbana, Ill.).

Bonnor, W. B. The stability of cosmological models. Z. Astrophys. **35** (1954), 10–20.

This paper is concerned with the joining of two universes U_- and U_+ across $t=0$. For $t<0$ we have the Friedmann non-static homogeneous model

$$U_-:\ ds^2=-e^{g(t)}[(1-r^2/R^2)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)+dt^2],$$

and for $t>0$ the general spherically symmetric model

$$U_+: ds^2=-e^{\lambda}dr^2-e^{\omega}(d\theta^2+\sin^2\theta d\phi^2)+e^{\nu}dt^2,$$

where λ, ω, ν are functions of r and t. The junction conditions are the continuity of g_{ik}, $\partial g_{\mu\nu}/\partial t$ and T_k^4 ($i, k=1, 2, 3, 4; \mu, \nu=1, 2, 3$). It is assumed that there is an isotropic pressure in U_+. It is shown that if the same equation of state $p=f(\varrho)$ holds for U_- and U_+, then U_+ is the continuation of U_-; if there are different equations of state in U_- and U_+, then there are discontinuities in p and its time derivatives, and U_+ is another, different, Friedmann universe; if no equation of state is assumed in U_+, then inhomogeneities may develop. *J. L. Synge.*

Shibata, Takashi. On Lorentz transformations and continuity equation of angular momentum in relativistic quantum mechanics. J. Sci. Hiroshima Univ. Ser. A. **18** (1955), 391–398.

The author defines a three-parameter subgroup of the Lorentz group as that subgroup which leaves the direction of a particular null-vector invariant. The method of choosing this null vector is unspecified. A Lagrangian function is assumed to be invariant under the restricted group described above and the restricted form of the conservation of angular momentum is derived as is the definition of the total angular momentum. *A. H. Taub.*

Halbwachs, Francis, Lochak, Georges, et Vigier, Jean-Pierre. Décomposition en fonction de variables dynamiques du tenseur d'énergie-impulsion des fluides relativistes dotés de moment cinétique interne. C. R. Acad. Sci. Paris **241** (1955), 692–695.

The author defines the spin angular momentum, the linear momentum and the "torsion" for a system whose field equations are derived from a variational principle involving a Lagrangian function depending on field variables and their first derivations alone. Equations relating these quantities are also given. These are deduced from the conservation laws satisfied by a non-symmetrical stress-energy tensor. *A. H. Taub* (Urbana, Ill.).

Grünbaum, Adolf. Logical and philosophical foundations of the special theory of relativity. Amer. J. Phys. **23** (1955), 450–464.

See also: Duschek und Hochrainer, p. 214.

MECHANICS

Hulubei, Dan. Sur les configurations relatives à l'équilibre astatique. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **6** (1955), 203–235. (Romanian. Russian and French summaries)

The rigid body (S) is said to be in astatic (indifferent) equilibrium under the action of n constant forces F_i ($i=1, 2, \cdots, n$), applied at the fixed points P_i of (S), provided that (S) remains in equilibrium regardless of its position. The author gives a proof for the (known) conditions for the astatic equilibrium and investigates some particular instances. The cases $n=1$, 2, 3 and that of 4 forces not in parallel planes are trivial. The general cases of 4, 5, 6 and 7 forces are not trivial and are studied in some detail. Nine theorems are proven,

connecting projectively the set of points P_i and the directions g_i of the F_i. Characteristic is the following statement: In the case $n=6$, if all points P_i belong to the same conic section, then the 6 directions g_i are the generatrices of a second-degree conic. *E. Grosswald.*

Egger, H. Zur Anwendung des Prinzips der virtuellen Verschiebungen auf die Ermittlung von Gleichgewichtslagen. Österreich. Ing.-Arch. **9** (1955), 86–93.

The problem is to determine the equilibrium configuration of systems of rigid bodies connected by frictionless joints or bearings. From the principle of virtual displacements some simple general formulae are derived. Their application is illustrated by solving several complex systems as examples. *H. D. Block.*

Freudenstein, Ferdinand. Approximate synthesis of four-bar linkages. Trans. A.S.M.E. **77** (1955), 853–859; discussion 859–861.

The function generated by a four-bar linkage has seven independent parameters. They may be taken as the three ratios of the lengths of the cranks and connecting-rod with respect to the center distance, the two scale factors of the crank rotations and the two zero positions of the cranks. The equations that must be satisfied for the function to be satisfied at three, four and five points are displayed. For accuracy in the neighborhood of a desired point, the equations for satisfying the position and the first four derivatives are displayed. Examples are given in which the first six derivatives are satisfied. *M. Goldberg.*

Weirich, H. Zur Ermittlung des Trägheitspoles und der Trägheitspolkurve. Österreich. Ing.-Arch. **9** (1955), 230–238.

The motion of a plane body over a fixed plane is derivable from a knowledge of its center of gravity, its radius of gyration and the applied forces. The characteristic poles and loci of the purely kinematic motions are discussed briefly. The dynamic parameters are derived by means of the inertia pole which is the center of a linear set of forces which produce the same angular velocity about the same center. A graphical construction is given for the determination of the inertia pole from the tangent pole or the impulse pole. Application is made to motions in which two points of the body are constrained to move along fixed paths. In particular, the motion of the cross-slide is examined analytically. *M. Goldberg* (Washington, D.C.).

Bottema, O. Note on a non-holonomic system. Quart. Appl. Math. **13** (1955), 191–192.

In order to illustrate non-holonomic equations of constraint, the sphere rolling on a rough plane has long been a standard example. A simpler example was provided by Caratheodory's "sleigh problem" [Z. Angew. Math. Mech. **13** (1933), 71–76]. The paper under review considers the case of a particle moving on a horizontal disk which is free to rotate about a vertical axis. As constraint the author takes the conservation of angular momentum and shows that the relation is non-integrable.

The reviewer believes that this is not a constraint at all, since the condition is not kinematical but dynamical and is actually a consequence of the unconstrained equations of motion and the initial conditions; and thus the system is in fact holonomic. It is true that the concept "non-holonomic" has not (to the reviewer's knowledge) been formulated in a precise way in the literature. To remedy this the reviewer would like to offer the following definition.

(1) Let S be a collection of statements (for example ordinary or differential equations or inequalities). (2) Let $\bar{S}$ denote the set of all statements implied by those in S. (3) Two systems of statements S and T are called "equivalent" ($S\sim T$) if $\bar{S}=\bar{T}$ (i.e. if each set implies the other). (4) Given a system of statements S (e.g. the Newton or Lagrange or Hamilton equations of motion and the initial displacements and velocities), a statement c is called a "constraint" on S if $\overline{S+c}\neq\bar{S}$ (i.e. $(c+S)\nsim S$.) (5) A set of constraints on S, C, is "holonomic" if $\bar{C}$ is equivalent to a system of equations of the form $F_i(q_1, \cdots, q_n)=0$, where F_i is a real-valued function of the n real variables $q_1, \cdots, q_n$ (the coordinates), (and not their differentials). (6) A system of constraints on S is "non-holonomic" if it is not holonomic.

The reviewer hopes that this captures the intention of the previous vague formulations.

There are some minor errors in the computations of Lagrange's equations, but these will easily be corrected by the reader. *H. D. Block* (Ithaca, N.Y.).

Mendes, Marcel. Equations de Lagrange, équations canoniques et généralisations. J. Math. Pures Appl. (9) **34** (1955), 145–183.

The various generalizations of Lagrange's and Hamilton's equations contained in this paper have been summarized by the author in four previous notes [C.R. Acad. Sci. Paris **233** (1951), 1574–1575; **234** (1952), 1665–1667; **235** (1952), 408–409; **236** (1953), 457–458; MR **14**, 100; **13**, 748; **14**, 171, 917]. *D. C. Lewis.*

Ghosh, N. N. Derivation of Lagrangian and Hamiltonian equations of motion from the dynamics in hyperspace. Bull. Calcutta Math. Soc. **46** (1954), 169–177.

Given a system of N particles subject to r independent holonomic constraints, one can regard the motion of the system as the motion of a point on a $(3N-r)$-dimensional manifold embedded in Cartesian $3N$ space. The author considers a base of vectors for the tangent subspace and its reciprocal set (some readers might prefer to interpret this as the dual base in the conjugate space); similarly for the osculating subspace of second order. He also introduces several operations which are similar to some used in the Exterior Differential Calculus. Newton's laws are applied to this geometrical apparatus, and the equations of Lagrange (with generalized forces) and Hamilton are derived. *H. D. Block* (Ithaca, N.Y.).

Agostinelli, Cataldo. Sui sistemi canonici che ammettono particolari classi di soluzioni stazionarie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88** (1954), 121–126.

A sufficient condition that ∞^{2n-m} solutions of the Hamiltonian equations

$$\dot{q}_r=\partial H/\partial p_r, \quad \dot{p}_r=-\partial H/\partial q_r, \quad r=1, \cdots, n,$$

can be found by solving the system $\partial H/\partial p_i=0$ ($i=1, \cdots, m<n$), together with the last $n-m$ of each group of the original equations, is given in the form of m identities to be satisfied by some of the first and second derivatives of H, at least in case the Hessian of H with respect to the first m p's is not zero. The first m q's in any such solution are, of course, constants. *D. C. Lewis.*

Grioli, G. Forma intrinseca delle equazioni dinamiche del solido pesante asimmetrico con un punto fisso e ricerca dei moti di precessione. Ann. Univ. Ferrara. Sez. VII. (N.S.) 3 (1954), 31–43.

So called intrinsic equations of the motion of a rigid body about a fixed point are obtained, reminiscent of those due to W. F. Osgood [Trans. Amer. Math. Soc. 23 (1923), 240–264] in the symmetric case of the gyroscope. These equations are used to study the existence of precessions in the case of asymmetric rigid bodies.
D. C. Lewis (Pacific Palisades, Calif.).

Vacca, Maria Teresa. Su un problema più generale di quello di De Brun per il moto di un corpo rigido intorno a un punto fisso. Boll. Un. Mat. Ital. (3) 10 (1955), 52–58.

This paper concerns the motion of a rigid body about a fixed point when each particle of the body (assumed in a given orientation) is acted upon by a force which is a certain linear function of the coordinates of the particle referred to principal axes fixed in the body. This linear function is quite special, but it generalizes the still more special case, treated by F. de Brun, in which the force on the particle is proportional to its distance from a plane fixed in space. The equations admit first integrals corresponding to conservation of energy and angular momentum. When a certain relationship is satisfied by the parameters of the system, one has a third first integral, which together with the other two permits complete integration in quadratures. *D. C. Lewis.*

Bankovskaya, N. V. Stability of the vertical position of the axis of a heavy gyroscope with harmonic oscillations of the point of support. Ukrain. Mat. Ž. 6 (1954), 418–422. (Russian)

A point O on the axis of a heavy symmetrical gyro is forced to perform a simple harmonic motion $z=z_0 \sin wt$. A motion at constant spin with the axis of figure vertical is possible. For the study of the stability of this motion, the Mathieu equation,
$$\ddot{u}+\lambda^2(1+\mu \sin wt)u=0$$
is derived, assuming a sufficiently small angle $\pi/2-\delta_2$ with the vertical, and putting $u=(\delta_1+i\delta_2) \exp(-iCs/2At)$ where s is the (constant) spin, δ_1 the angle of precession, C the axial, and A the equatorial moment of inertia. The constant μ is proportional to $1/\lambda^2$, while
$$\lambda^2=(Cs/Aw)^2-4M/w^2,$$
M being the magnitude of moment of the weight about O.

The usual (Poincaré-Lindsted-Lyapunov) method of successive periodic approximations, based on expansions of λ and u into power series in μ, is applied [cf., e.g., Stoker, Nonlinear vibrations, Interscience, New York, 1950, p. 208; MR 11, 666]. Regions of instability are determined from the second approximation in the form
$$|\lambda^2-1|<\mu, \ |\lambda^2-4-2\mu^2/3|<\mu^2;$$
for the cases when for $\mu=0$ the values of λ^2 are, respectively, 1 and 2. These cases correspond to certain two values of the spin of the undisturbed motion, distinguished, as far as this reviewer can see, only by their docility. The brief last section deals obscurely with the behavior of the unstable solutions when s tends toward zero, and probably could be interpreted if necessary.
A. W. Wundheiler (Chicago, Ill.).

Garnier, Maurice. Points remarquables sur une trajectoire. Mém. Artill. Franç. 24 (1950), 953–1008; 25 (1951), 7–76, 691–741, 745–804; 26 (1952), 625–684; 27 (1953), 513–609.

This paper covering altogether 383 pages discusses the computation of various significant points on a trajectory, each usually an extremal in some respect, such as with respect to altitude, velocity, acceleration, or a point of contact or of indetermination. The methods involve linear and quadratic interpolation to account for determined deviations from standard conditions. The author provides 752 model blank forms to effect the computation but does not here furnish numerical (doubtless confidential) information. The theory is that of a particle pursuing a course exactly determined by a known differential equation. The contribution here is toward such detailed organization of hand computation in certain special ballistic problems, that an unskilled computer with a taste for numbers and provided with appropriate tables and observed data may prepare ballistic tables. *A. A. Bennett* (Providence, R.I.).

Darpas, J.-G. Le mouvement du projectile autour de son centre de gravité. Mém. Artill. Franç. 27 (1953), 857–946.

This is a 90-page presentation of a course of lessons given to a class in naval artillery to supplement lectures based on the particle trajectory. It describes practical methods of observation and computation of yaw. It is rendered concrete by 28 figures, some with explicit numerical information based on experiment and 131 explicit formulas. The paper discusses stability not only of spinning but also of fin-stabilized projectiles and of rockets. *A. A. Bennett* (Providence, R.I.).

See also: Ščigolev, p. 198; Moiseev, p. 198; Duschek und Hochrainer, p. 214.

Fluid Mechanics, Acoustics

Yih, Chia-shun. On a method of generating three-dimensional potential flows from two-dimensional ones. Quart. Appl. Math. 13 (1955), 320–322.

Les écoulements dont il s'agit sont ceux d'un fluide parfait incompressible admettant un potentiel des vitesses. La méthode repose sur la remarque suivante: $\phi=f(x, y)+g(y, z)+h(z, x)$ est une fonction harmonique si les fonctions f, g, h sont des fonctions harmoniques de 2 variables. *P. Germain* (Paris).

Consiglio, Alfonso. Moti piani irrotazionali di liquidi perfetti nei campi semplicemente connessi. Matematiche, Catania 9 (1954), 113–121.

The author finds the relations which connect the Fourier coefficients of two series which satisfy identically $X(\theta) \cos \theta+Y(\theta) \sin \theta=0$ and remarks that this result can be used to express the boundary conditions when a perfect liquid moves in the presence of a circular boundary. The results are applied to the uniform motion of a circular cylinder. This and all other similar problems can, however, be solved without calculations by the reviewer's Circle Theorem [Theoretical hydrodynamics, 2nd ed., Macmillan, New York, 1950, § 6–21; MR 11, 471].
L. M. Milne-Thomson (Greenwich).

Hunt, J. N. Gravity waves in flowing water. Proc. Roy. Soc. London. Ser. A. **231** (1955), 496–504.

Two-dimensional gravity waves on a non-uniform current are considered. For linear waves on a small current varying as the one-seventh power of distance from the bottom, the wave velocity is found as a power series in a Froude number. Non-linear solitary and cnoidal waves are determined to a first approximation. No numerical results are given. *F. Ursell.*

Litwiniszyn, Jerzy. Differential-integral equations of motion of a deformable fluid mass bounded by a surface subjected to stresses. Arch. Mech. Stos. **5** (1953), 15–20. (Polish. English summary)

Les équations mentionées dans le titre sont dérivées par moyen du principe de moments et en changeant les variables euleriens en variables lagrangeiens. De cette manière l'auteur obtient un région d'intégration indépendant du temps. La forme finale des équations du mouvement est obtenue en vertue d'un théorème de Lindelöf [voir Kamke, Differentialgleichungen, Bd. I, 2te Aufl. Akademische Verlagsgesellschaft, Leipzig, 1943; MR **9**, 587]. L'auteur expose l'opinion que le théorème ordinaire de Gauss-Green ne peut être appliqué, parce que les fonctions des committants du tenseur métrique de surface sont déterminées seulement pour la surface et non pour la région spatiale limité par lui. L'auteur ne cite aucun travail dans cette domaine.
K. Bhagwandin (Oslo).

Heinrich, G., und Desoyer, K. Hydromechanische Grundlagen für die Behandlung von stationären und instationären Grundwasserströmungen. Ing.-Arch. **23** (1955), 73–84.

The authors derive differential equations for groundwater flow, and state the various boundary-conditions [G. Hamel, Z. Angew. Math. Mech. **14** (1934), 129–157, p. 131; H. F. Rossbach, Ing.-Arch. **12** (1941), 221–246, p. 222; MR **3**, 287] pertaining to them. The various stages in the development of the equations are discussed in detail, and it is shown that their theory is more general than the usual Darcy-type approximations. Numerical results are not presented. However, the authors promise to do so in a later communication, where relaxation methods will be resorted to.

It is to be regretted, however, that the authors do not seem to be familiar with the fundamental contributions of N. E. Žukovskiĭ [Ž. Russkogo Fiz.-Him. Obšč. Č. Fiz. **21** otd. 1, vyp. 1, 1–20 (1889)=Izbrannye sočineniya, v. 1, pp. 153–170, Moscow 1948], N. N. Pavlovskiĭ [Neravnomernoe dviženie gruntovyh vod, Kubuč, Leningrad, 1932], and the more recent contributions of P. Ya. Polubarinova-Kočina [e.g., Nekotorye zadači ploskogo dviženiya gruntovyh vod, Akad. Nauk SSSR. Moscow, 1942]. Numerous additional references to problems connected with ground-water flow are to be found in these reviews. *K. Bhagwandin* (Oslo).

Gibellato, Silvio. Strato limite attorno ad una lastra piana investita da un fluido incompressibile dotato di una velocità che è somma di una parte costante e di una parte alternata. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1955), 180–192.

The author studies the two-dimensional laminar boundary layer on a flat plate in a stream which ahead of the plate is parallel to it and has the fluctuating speed $U_\infty+u_\infty e^{i\omega t}$. The ratio u_∞/U_∞ is considered so small that the resulting perturbations of the Blasius boundary-layer equation can be linearised. The linear perturbation equation is then solved by expansion in powers of ω, the coefficients of 1, ω and ω^2 being determined by numerical solution of the appropriate ordinary differential equations. Finally the corresponding terms in the series for the fluctuating parts of the skin friction and displacement thickness are determined.

The reviewer naturally wished to compare the accurately determined coefficients of ω and ω^2 with his own approximate theory [Proc. Roy. Soc. London. Ser. A. **224** (1954), 1–23; MR **15**, 907] (the coefficient of 1 is trivially derived from the Blasius solution and is the same in both theories). The coefficients of ω agree to within 1 per cent. The coefficient of ω^2 as calculated by the author produces a maximum change of 30 percent (in the in-phase component of skin friction) at the frequency ω_0 above which the reviewer recommended changing over to the high-frequency (shear-wave) approximation, while recommending the neglect of ω^2 below it. *M. J. Lighthill* (Manchester).

Meksyn, D. Integration of the boundary-layer equations for a plane in compressible flow with heat transfer. Proc. Roy. Soc. London. Ser. A. **231** (1955), 274–280.

The author extends his asymptotic method of integration of the compressible boundary-layer equations [same Proc. **195** (1948), 180–188; MR **10**, 336] to the general case of heat transfer and arbitrary Prandtl number (close to unity). Zero free-stream pressure gradient, constant wall temperature, and a power-law variation of viscosity and thermal conductivity with absolute temperature are assumed. For the velocity and temperature distributions and the skin-friction and heat-transfer coefficients simple expressions are obtained, which show clearly their dependence on the physical parameters. When the temperature of the wall is either above, or not much below, that of the free stream, the results agree closely with Crocco's numerical computations. *D. W. Dunn* (Baltimore, Md.).

Li, Ting Yi. Simple shear flow past a flat plate in a compressible viscous fluid. J. Aero. Sci. **22** (1955), 724–725.

Li, Ting-Yi, and Nagamatsu, Henry T. Similar solutions of compressible boundary-layer equations. J. Aero. Sci. **22** (1955), 607–616.

In the case of gases whose Prandtl number is equal to unity and whose viscosity coefficient increases linearly with temperature, it was known that the compressible boundary-layer equations can be transformed into a system of total differential equations for a special velocity distribution in the main stream. By a direct procedure, the differential equation defining this velocity distribution is obtained explicitly. With this form of solutions, the authors intend to study (a) the heat-transfer characteristics under various surface temperatures and different pressure gradients and (b) the effects of heat transfer on the phenomenon of laminar separation.
Y. H. Kuo (Ithaca, N.Y.).

Mawardi, Osman K. On the spectrum of noise from turbulence. J. Acoust. Soc. Amer. **27** (1955), 442–445.

This paper consists mainly of a review of the theory of sound generated aerodynamically. In the last section, however, an attempt is made to calculate the spectrum

of the noise generated by eddies of a given size, by assuming the auto-correlation for a product of two turbulent velocity components to be 1 within a sphere of the given diameter and 0 outside. Several points made in this section seem dubious to the reviewer.

It is alleged that it is principally eddies of wavenumer ω/c that generate sound of frequency ω (where c is the velocity of sound). Apart from the evident point that the spectrum of a velocity-product, which contains summation and difference tones of the turbulence spectrum itself, is relevant, one can question the conclusion on the more positive ground that the reviewer's limiting form of the radiation field for low Mach number [Proc. Roy. Soc. London. Ser. A, **222** (1954), 1–32, p. 12, eqn. (6); MR **15**, 837] does not bring in c at all except as a factor outside the integral. The same is true of Proudman's more detailed calculation for isotropic turbulence [ibid. **214** (1952), 119–132; MR **14**, 596], and it is easily seen that the same must be true of the frequency spectra, which are given by the same formula with T_{ij} replaced by its Fourier transform.

The two points mentioned provide the clue to the mystery. The author's ω/c can to a good approximation be replaced by 0, so that the sound of frequency ω which is radiated depends on the spectral density of the velocity-product for frequency ω and wave-number zero. This density is made up entirely of difference tones (as far as wave-number is concerned) and eddies of all wave-numbers contribute to it.

Actually, eddies of wave number ω/u are likely to be responsible for sound of frequency ω, where u is a scale of the velocity fluctuation. This is why the high-frequency end of the curve of quadrupole radiation against frequency is not likely to be of much importance for turbulence of low root-mean-square Mach number (although in the theory of the paper under review it plays a prominent part).

A less serious point is that the author appears to use the wrong formula for the strength of a source made up of three equal mutually orthogonal longitudinal quadrupoles, and hence gets a spectral dependence on the square of the frequency for low frequencies, whereas an essentially quadrupole source of sound like turbulence must, as the reviewer showed in the paper cited, have acoustic power proportional to the fourth power of the frequency at low frequencies. *M. J. Lighthill.*

Kraichnan, Robert H. On the statistical mechanics of an adiabatically compressible fluid. J. Acoust. Soc. Amer. **27** (1955), 438–441.

The paper represents an attempt to apply the methods of the statistical mechanics of classical conservative systems, particularly Liouville's theorem and the H-theorem, to problems of fluid flow involving turbulence and random noise. The author shows that a form of Liouville's theorem can be obtained for an enclosed compressible flow in which viscosity and heat conduction are neglected. This is because such a system is conservative (a fact of which Kelvin's circulation theorem is the classical expression), so that the representation of the flow by means of generalised coordinates as in the works of Kelvin and Rayleigh leads at once to the desired results.

The author goes on to infer, in the case when the root mean square Mach number is small, a principle of equipartion of energy, not only between acoustic and turbulent modes (which would be credible) but also between modes of different wave numbers. The latter is hardly credible in view of the known tendencies of both sound and turbulence to pass energy continually down to large wave numbers, a tendency present whether or not viscous effects are acting as a sink of energy at the high-wave-number end of the spectrum.

The reviewer has tried to find the source of the apparent discrepancy, which is difficult as the author's whole argument is confined to two sentences: "According to the H-theorem, which then [from Liouville's theorem] follows, an ensemble of systems will tend toward an equilibrium distribution uniform over the entire region of phase space permitted by the constants of motion. In the case of a non-viscous fluid confined in a large box, the only constant of motion is the total energy." Two points may be worth following up. (i) The system considered is possible for a finite time only; notoriously, sound waves could exist in such a medium only for a finite time, after which discontinuities would appear; and various theoretical investigations [see, e.g., Proudman and Reid, Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 163–189; MR **16**, 299] have indicated that the same would probably happen in vortex motions (showing, for example, that the mean square vorticity would become infinite). (ii) In most forms of the H-theorem it is assumed that the H integral is bounded below, which may possibly not be true for systems where the energy can catastrophically reach infinite wave-number in a finite time.

The author goes on to discuss, in a more realistic section which shows some affinity with the work of Moyal [Proc. Cambridge Philos. Soc. **48** (1952), 329–344; MR **14**, 106], the possible effect of viscosity on his results, particularly in connection with the cutoff which it introduces at high wave numbers. Finally, he refers briefly to future work in which the methods he has developed will be applied to non-equilibrium situations.

M. J. Lighthill (Manchester).

Saïssac, J. Sur la diffusion turbulente homogène des particules. J. Sci. Météorol. **7** (1955), 227–232. (English and Spanish summaries)

Steketee, J. A. Note on a formula of H. W. Emmons. J. Aero. Sci. **22** (1955), 578–579.

Démonstration simplifiée d'une formule de probabilités utilisée par H. W. Emmons [même J. **18** (1951), 490–498; MR **14**, 697]. *J. Bass* (Paris).

Woods, L. C. Compressible subsonic flow in two-dimensional channels. I. Basic mathematical theory. Aero. Quart. **6** (1955), 205–220.

Le fluide compressible, dont on étudie l'écoulement dans un canal, est un fluide fictif de Chaplygin-Kármán-Tsien, c'est à dire que l'étude des écoulements d'un tel fluide se ramène essentiellement à celle des écoulements d'un fluide incompressible. Le travail repose sur un théorème donnant la valeur d'une fonction analytique définie dans une bande, en fonction des valeurs prises par sa partie imaginaire sur les deux droites parallèles limitant la bande. La variable utilisée est le potentiel complexe w, la fonction, la vitesse complexe (généralisée) f; de la connaissance de $f(w)$, on déduit l'écoulement dans le plan physique. Dans le problème direct, les parois du canal sont données; la résolution ne peut se faire que par des procédés d'itération qui restent laborieux. Le problème inverse, pour lequel les parties réelles

ou imaginaires de f sont connues sur les parois est immédiat. Si on fait croître indéfiniment la largeur du canal, les résultats obtenus redonnent à la limite ceux bien connus de la théorie des profils. Il semble qu'un des précédents mémoires de l'auteur [Quart. J. Mech. Appl. Math. **7** (1954), 263-282; MR **16**, 418] envisageait déjà des problèmes analogues mais d'un caractère plus général.
P. Germain (Paris).

Bernstein, B. and Thomas, T. Y. The differential equations of the stream lines for compressible gas flow. J. Rational Mech. Anal. **4** (1955), 703–719.

L'écoulement est plan, le fluide parfait, l'enthalpie génératrice est constante, mais l'entropie peut varier d'une ligne de courant à l'autre. A partir des équations de l'écoulement, on élimine d'abord la pression et la densité pour ne garder comme fonctions inconnues que les composantes de la vitesse. Les auteurs introduisent alors les arcs des lignes de courant et de leurs trajectoires orthogonales et prennent comme fonctions inconnues l'angle du vecteur vitesse avec une direction fixe ω et la quantité $G=m^2-1$, m désignant le nombre de Mach local. Ils forment les 2 équations définissant ω et G. Eliminant alors les dérivées de G, on obtient une équation du troisième degré en G, dont les coefficients sont fonctions de ω et de ses dérivées. Le cas particulier où tous les coefficients de cette équation sont identiquement nuls est caractéristique d'un écoulement de source. Ecartant ce cas, les auteurs forment les équations (non indépendantes) définissant la fonction ω et discutent alors la solution de certains problèmes aux limites; le cas où l'on connait à priori une ligne de courant est spécialement envisagé.
P. Germain (Paris).

Lidov, M. L. On the theory of solutions linearized about one-dimensional self-similar motions of a gas. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1089–1092. (Russian)

In the general case of unsteady adiabatic motion of a perfect gas, referred to spherical polar coordinates, the velocity components, pressure and density can be written, with an obvious notation, in the form,

$$V_r=b^{1/m}t^{-(n/m)-1}\tilde{V}_r(\lambda,\nu,\theta,\phi),\ \text{etc.},$$
$$P=ab^{-(k+1)/m}t^{(n/m)(k+1)-(s+2)}\tilde{P}(\lambda,\nu,\theta,\phi),$$
$$\varrho=ab^{-(k+3)/m}t^{(n/m)(k+3)-s}\tilde{R}(\lambda,\nu,\theta,\phi),$$

where $\lambda=br^{-m}t^{-n}$, and $\nu=et^{l}$ are dimensionless variable parameters, $\tilde{V}_r$, $\tilde{V}_\theta$, $\tilde{V}_\phi$, P and $\tilde{R}$ dimensionless functions.

It is assumed that for $\nu=0$ the motion degenerates into a self-similar motion with spherical symmetry so that $[\tilde{P}]_{\nu=0}=P_0(\lambda)$, etc. The effect of a small disturbance to a self-similar motion is then represented by considering the general motion when ν is small. The problem can be reduced to that of solving a system of 5 equations, linear in derivatives with respect to λ,θ,ϕ and with coefficients depending on the basic exact solution.

Using dimensional arguments, integrals of this system are obtained in the following two cases. (1) When the perturbed motion is not self-similar, but has spherical symmetry. (2) When the dependent variables can be written in the separable form

$$P(\lambda,\theta,\phi)=P_1(\lambda)P_2(\theta,\phi),\ \text{etc.}$$

Complete details are not given. The author states that other cases can be worked out and that the work can be extended to discuss small deviations from self-similar motions with cylindrical or plane symmetry. *M. Holt.*

Lidov, M. L. A finite integral of the equations of one-dimensional self-similar adiabatic motions of a gas. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 35–36. (Russian)

Self-similar solutions of the equations of one-dimensional unsteady gas flow are considered, with account taken of gravitational forces. In the general case a motion is self-similar provided the velocity, density, pressure and mass can be written

$$u=b^{1/m}t^{-(n/m)-1}V_1(\lambda),$$
$$\varrho=ab^{-(k+3)/m}t^{(n/m)(k+3)-s}R_1(\lambda),$$
$$P=ab^{-(k+1)/m}t^{(n/m)(k+1)-(s+2)}P_1(\lambda),$$
$$\overline{m}=ab^{-(k+3-\nu)/m}t^{(n/m)(k+3-\nu)-s}M_1(\lambda)$$

respectively, where λ is a dimensionless argument and V_1, R_1, P_1, M_1 satisfy a system of 4 linear ordinary differential equations. The author uses a dimensional argument due to Sedov [Similarity and dimensional methods in mechanics, Gostehizdat, Moscow-Leningrad, 1954; for a review of the 1951 edition see MR **14**, 809] to deduce a first integral of this system which is independent of boundary and initial conditions. *M. Holt.*

Roper, G. M. Some applications of the Lamé function solutions of the linearised supersonic flow equations. Aero. Res. Council, Rep. and Memo. no. **2865** (1951), 42 pp. (1955).

Ce travail contient 2 parties distinctes. Dans la première, l'auteur donne les cartes de pression sur des ailes dont le bord d'attaque est subsonique; il en déduit les valeurs des trainées (en tenant compte de la contribution due aux singularités du bord d'attaque). Les solutions utilisées par l'auteur sont obtenues à l'aide de fonctions de Lamé. On comprend peut être mieux le succès du calcul et la simplicité du résultat en notant qu'il s'agit d'ailes obtenues par superposition de mouvements homogènes conduisant à des répartitions de pression qui sont des polynomes en fonction des variables cartésiennes définissant l'aile. Ainsi que l'a montré le reviewer [Rech. Aéro. no. **7** (1949), 3–16; MR **10**, 492], al détermination de tels écoulements peut s'effectuer de façon très élémentaire.

La deuxième partie est relative à des écoulements antisymétriques ou portants. On y étudie l'effet de la courbure et du vrillage. Il s'agit encore de superposition d'écoulements homogènes particuliers. Les applications faites dans chaque partie sont clairement présentées, et les résultats obtenus sont très instructifs. *P. Germain.*

Bévierre, P. Détermination d'un écoulement compressible subsonique à partir d'un écoulement incompressible. Rech. Aéro. no. **46** (1955), 3–5.

Use is made of an approximate compressibility law to establish a correspondence between a subsonic compressible flow and a given incompressible flow. The transformations are made without use of the hodograph method. The law is obtained as the condition that the equation for the stream function in compressible flow, after transformation of coordinates, reduces to Laplace's equation. The law, relating mass flow with Mach number, contains two arbitrary parameters. These are chosen to give agreement with the exact law (1) at the free stream Mach number and at the stagnation point (2) so that the two curves touch at the free stream Mach number. The results are applied to thin subsonic aerofoils at incidence.

A second transformation is developed in terms of the

potential function and can be applied to three-dimensional wings without lift. This is equivalent to a local Prandtl-Glauert transformation. *M. Holt* (Providence, R.I.).

Fox, Phyllis A. Perturbation theory of wave propagation based on the method of characteristics. J. Math. Phys. 34 (1955), 133–151.

The reviewer's technique for rendering approximate solutions to physical problems uniformly valid was applied by him [Philos. Mag. (7) **40** (1949), 1179–1201; MR **11**, 518] and by Whitham [see, e.g., Comm. Pure Appl. Math. **5** (1952), 301–348; MR **14**, 330] to a number of problems involving partial differential equations of hyperbolic type. The technique involves the choice of new independent variables, and the expansion of the original independent variables in a series whose coefficients are functions of the new ones. In the classes of problem treated by the reviewer and by Whitham, the approximate solution became uniformly valid if one of the new independent variables was chosen as a "characteristic variable", in the sense of the theory of hyperbolic equations. The choice of the other variable was found not to be crucial, and it was most convenient in practice to retain one of the original independent variables for this purpose.

The present paper demonstrates rigorously that the technique will always be effective if both new independent variables are taken as characteristic variables. Though it is not implied that this will always be the most convenient choice, two examples are given for which such a change of both independent variables is necessary if the approximate solution is to be rendered uniformly valid. A solution is thus found to the problem of the motion produced when a gas initially held at rest, with uniform entropy and a sound velocity varying sinusoidally with one of the coordinates, is allowed to move freely. The positions of shock formation are found, although the author does not follow Whitham so far as to use the technique to calculate the development of the shock waves beyond their initial formation. The motion produced in a semi-infinite cylinder by periodic oscillation of the piston at one end is similarly treated; a fuller solution of this problem was given by Chester [Quart. J. Mech. Appl. Math. **5** (1952), 408–422; MR **14**, 548]. The general approach of the author shows considerable affinity with the work of Meyer [ibid. **5** (1952), 257–269, 270–291; MR **14**, 329]. *M. J. Lighthill.*

Latter, Richard. Similarity solution for a spherical shock wave. J. Appl. Phys. **26** (1955), 954–960.

L'auteur étudie les mouvements d'un fluide compressible, visqueux, non conducteur de chaleur, dans le cas de la symétrie sphérique. La pression étant designée par p et les efforts intérieurs par $p+q$, l'expression adoptée pour le terme q dû à la viscosité est la suivante:

$$2q=K^2\varrho r^2(\partial u/\partial r)\{|\partial u/\partial r|-(\partial u/\partial r)\},$$

K désignant une constante, u la vitesse du fluide et r la distance au centre des coordonnées sphériques. Pour la valeur $K=0$, les équations du mouvement sont intégrées analytiquement et on retrouve la solution de Taylor [Proc. Roy. Soc. London Ser. A. **201** (1950), 159–174]; l'intégration a ensuite été effectuée numériquement pour trois valeurs non nulles de K. La comparaison des résultats fait apparaître que l'influence de la viscosité est très importante.

Dans l'étude des phénomènes de viscosité, on a l'habitude d'adopter la loi $q=-\frac{4}{3}\mu(\partial u/\partial r)$ avec $\mu>0$. Le coefficient μ de l'auteur possède la valeur

$$\mu=\tfrac{3}{8}K^2\varrho r^2\{(\partial u/\partial r)-|\partial u/\partial r|\},$$

laquelle est toujours négative ou nulle. *H. Cabannes.*

Taub, A. H. Determination of flows behind stationary and pseudo-stationary shocks. Ann. of Math. (2) **62** (1955), 300–325.

Ce mémoire est une étude analytique locale très complète des écoulements au voisinage d'un choc, dans le cas où l'écoulement est stationnaire ou pseudostationnaire (la configuration de l'écoulement reste, quand le temps varie, homothétique à elle même). Pour simplifier cette analyse nous ne parlerons que du cas stationnaire. Les lignes de coordonnées choisies sont les lignes de courant et les lignes lieux de particules M qui ont mis le même temps pour atteindre la position M après avoir traversé le choc. Les équations du mouvement et les lois des chocs sont écrites avec les variables ainsi introduites. L'auteur étudie systématiquement les relations entre les propriétés géométriques de l'onde de choc et celles d'une ligne de courant (tangente, courbure, et dérivées d'ordre supérieur). Si l'écoulement amont est donné et si le choc est connu, la détermination de l'écoulement aval s'obtient par la résolution d'un problème de Cauchy. Si l'on connait une ligne de courant, l'écoulement sera déterminé (localement) si l'on parvient à déterminer le choc; ce problème fait l'objet d'une discussion détaillée; on retrouve en particulier, dans le cas où l'écoulement en aval du choc est subsonique, les circonstances singulières déjà signalées dans des cas particuliers par H. Cabannes [C.R. Acad. Sci. Paris **233** (1951), 354–356; MR **13**, 180]et T. Y. Thomas [J. Math. Phys. **28** (1949), 153–172; MR **11**, 479]. Le mémoire se termine par l'étude de chocs présentant des singularités (discontinuité de la tangente, de la courbure, ou d'une dérivée d'ordre supérieur); de tels chocs jouent un role important dans certains écoulements transsoniques et dans le phénomène de la réflexion de Mach. *P. Germain* (Paris).

Brillouin, J. Réflexion et réfraction d'ondes acoustiques par une onde de choc. Acustica **5** (1955), 149–163.

A mathematical analysis is made of phenomena accompanying the collision of a plane progressive acoustic wave with a shock wave. Taking into account viscosity and thermal conductivity, it is shown that a stationary thermal wave, as well as a refracted or reflected acoustic wave, is formed. The thermal wave decays slowly and aperiodically. The author suggests aerodynamicists might use the sound wave method for experimental studies of shock wave evolution. *W. W. Soroka* (Berkeley, Calif.).

Simon, R. Etude de la propagation des ondes dans le modèle de Roche généralisé. Ann. Astrophys. **18** (1955), 92–99.

A spherical mass of gas is considered in which shock-waves are propagated according to a linearised theory. The motion is such that each element of gas conserves its entropy. The material outside the shock-front is regarded as being in equilibrium, and the shock-wave is proved to move with the speed of sound appropriate to each position in the equilibrium configuration. The solution of the basic equation for the wave-disturbance is obtained by the Laplace transform method. Detailed consideration is given to waves in a gaseous envelope, surrounding a non gaseous core, the waves being started by the pistonlike

action of the surface of the core. The gravitational self-attraction of the envelope is neglected, but the gravitational force due to the central core is included. The equilibrium density in the envelope is proportional to $r^{-\alpha}$, where r is the distance from the center and $\alpha>0$. The solution takes a particularly simple form — in terms of a Hankel function — when the parameter

$$\tfrac{1}{2}\{(\alpha+4)^2-16(\alpha+1)/\gamma\}^{\frac{1}{2}}$$

has the value $\frac{1}{2}$, γ being the ratio of specific heats. [For a non-linear treatment of an analogous problem see Z. Kopal, Astrophys. J. **120** (1954), 159–171; MR **16**, 182.]
G. C. McVittie (Urbana, Ill.).

Fraenkel, L. E. The theoretical wave drag of some bodies of revolution. Aero. Res. Council, Rep. and Memo. no. **2842** (1951), 26 pp. (1955).

Le présent rapport n'apporte guère de résultats théoriques nouveaux. On y trouvera par contre des applications, poussées jusqu'au calcul numérique, de diverses théories proposées pour le calcul de la trainée des corps de révolution en présence d'un écoulement supersonique stationnaire. La comparaison et la discussion des résultats seront intéressantes pour les utilisateurs. *P. Germain.*

von Baranoff, A. Sur la résistance d'un corps de révolution en mouvement accéléré ou décéléré. Rech. Aéro. no. **45** (1955), 11–14.

This paper presents the derivation and details of the author's previously published extension [C.R. Acad. Sci. Paris **240** (1955), 591–593; MR **16**, 642] of the reviewer's results [J. Aero. Sci. **21** (1954), 644–645; MR **16**, 194] for transonic drag on an accelerated body of revolution. The author's result that the supersonic drag of a closed body could exceed its final (supersonic steady flow) value was criticized by the reviewer [MR **16**, 642] as being in numerical disagreement with the results of Wong [M.S. Thesis, Univ. of Calif., Los Angeles, 1955]; moreover, the reviewer had inferred that, in consequence of his symmetry theorem [loc. cit. ante], the author's results would imply negative drag for a decelerated body in some range of subsonic speeds and had regarded this as impossible. A numerical error recently has been found in Wong's results, however, while the author has informed the reviewer that his results have been subjected to an independent check. It appears, therefore, that the subsonic drag of a decelerated body would be negative.
J. W. Miles (Los Angeles, Calif.).

Belocerkovskiĭ, S. M. Spatial unsteady motion of a lifting surface. Prikl. Mat. Meh. **19** (1955), 410–420. (Russian)

A solution is given to the problem of unsteady motion of a lifting surface of general plan form with small curvature at small angles of incidence in an ideal incompressible fluid. The method of approach is based on an earlier investigation by the author [Prikl. Mat. Meh. **19** (1955), 159–164; MR **16**, 1060] of the velocity field due to a horseshoe vortex in unsteady flow. *M. Holt.*

Kafka, Paul G. Lifting pressure on delta wings with subsonic leading edges, symmetrical plan form, and discontinuous slope. J. Aero. Sci. **22** (1955), 725–726.

Van Dyke, Milton D. A study of hypersonic small-disturbance theory. NACA Rep. no. **1194** (1954), ii+21 pp. (1955).

This admirably comprehensive and well-documented account of hypersonic small-disturbance theory begins with a thorough analysis of the reduced equations of motion, the similarity rule, and the correspondence with unsteady flow in one less dimension, which are valid in the limit when the maximum angle τ between the surface of a body and the direction of motion becomes small, subject to $(M\tau)^{-1}$ being bounded, where M is the Mach number of its steady or unsteady motion through the air. Next it is shown how, if M is changed into $(M^2-1)^{\frac{1}{2}}$ at several places where it occurs in the equations and similarity rule, they become valid expressions of linearised supersonic flow theory with an error no greater than that which is in any case present in that theory. Conversely, in the hypersonic regime, the changes involve errors expressible by factors $1+O(\tau^2)$, which again were present already in the hypersonic small-disturbance theory. Finally, since the domains of validity of the two approximate theories can be shown to overlap, it follows that the modified equations and similarity rule are valid for small τ for all supersonic Mach numbers, down to those (very near 1) for which "transonic" effects first begin to appear. This rule gives already excellent agreement with the pressure on a wedge of 10° semi-angle over the whole range of M. In the case of a cone the agreement is tolerable even for 15° semi-angle.

The author makes it clear that it is the similarity rule which will be most useful to practical aerodynamicists, particularly for making use of experimental results. He gives also, however, a number of valuable predictions obtained by directly solving the reduced equations. These include the dependence of shock-wave curvature and surface pressure gradient on aerofoil curvature near the sharp leading edge of a two-dimensional aerofoil, for which a comparison with exact calculations is possible. The pressure gradient is again predicted well by the small-disturbance theory for a leading edge of 10° semi-angle. The shock-wave curvature is predicted really well only for 5° semi-angle, but for $\tau(M^2-1)^{\frac{1}{2}}>1$ both are predicted well for semi-angles even up to 15°.

A similar and even fuller treatment is given of conditions near the nose of an ogive of revolution, where a large amount of information that is completely new is obtained. The rather complicated singular behaviour near the body surface is very fully analysed, and the initial curvature as well as the initial gradient of the pressure distribution is determined numerically for a number of values of $\tau(M^2-1)^{\frac{1}{2}}$ and arbitrary ogive geometry.

Finally, an extremely full and valuable critical review of theories of hypersonic flow which claim to be applicable under conditions when the disturbance is not small is given, and in each case the theory is compared with the small-disturbance theory in the limit of small disturbances. *M. J. Lighthill* (Manchester).

Roberts, P. H. Hydromagnetic disturbances in a fluid of finite conductivity. Astrophys. J. **122** (1955), 315–326.

The equation

$$\frac{\partial}{\partial t}\left[\frac{\partial \mathbf{a}}{\partial t}-\frac{1}{4\pi\sigma}\nabla^2\mathbf{a}\right]=V^2\frac{\partial^2\mathbf{a}}{\partial z^2}\ \left(V^2=\frac{H_0^2}{4\pi\varrho}\right)$$

governs small disturbances in an unbounded, incompressible, inviscid fluid of electrical conductivity σ. In the undisturbed state the fluid is at rest, and a uniform magnetic field $\mathbf{H}_0$, parallel to the z-axis, is present. The vector $\mathbf{a}$ in the above equation may represent any of the following quantities: the disturbance magnetic field; the disturbance electric current; the fluid velocity; the

fluid vorticity. The discussion is confined to disturbances that depend on z and t only. The author treats some physical aspects of periodic disturbances (reflection and refraction, 'streak effect', skin depth), and uses operational techniques to derive a mathematical description of aperiodic disturbances. A previous discussion of aperiodic disturbances, by Walén [Ark. Mat. Astr. Fys. **30A** (1944), no. 15], neglected certain effects arising from the finite electrical conductivity. *D. Layzer* (Cambridge, Mass.).

See also: Chuang, p. 162; Magnus, p. 165; Khare, p. 199; Duschek und Hochrainer, p. 214.

Elasticity, Plasticity

Rivlin, R. S. Further remarks on the stress-deformation relations for isotropic materials. J. Rational Mech. Anal. 4 (1955), 681–702.

It has been shown by Rivlin and Ericksen [same J. 4 (1955), 323–425; MR **16**, 881] that general stress-deformation relations for isotropic materials can be reduced to isotropic relations of the form

$$\mathbf{T}=\mathfrak{F}(\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n),$$

where $\mathbf{T}$ and $\mathbf{A}_i$ are symmetric 3×3 matrices. The author investigates the possibility of rendering such relations into a more explicit form in the case when the components of $\mathbf{T}$ are polynomials in the components of the $\mathbf{A}_i$. His principal tools are the Hamilton-Cayley theorem and an extension of it. Most important result: A relation $\mathbf{T}=\mathfrak{F}(\mathbf{A}, \mathbf{B})$ is necessarily of the form

$$\mathbf{T}=\alpha_0\mathbf{I}+\alpha_1\mathbf{A}+\alpha_2\mathbf{B}+\alpha_3\mathbf{A}^2+\alpha_4\mathbf{B}^2+\alpha_5(\mathbf{AB}+\mathbf{BA}) +\alpha_6(\mathbf{A}^2\mathbf{B}+\mathbf{BA}^2)+\alpha_7(\mathbf{AB}^2+\mathbf{B}^2\mathbf{A})+\alpha_8(\mathbf{A}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{A}^2),$$

where the α's are polynomials in the 10 simultaneous invariants of $\mathbf{A}$ and $\mathbf{B}$ given by the traces of the following 10 matrices: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{A}^3$, $\mathbf{B}^3$, $\mathbf{AB}$, $\mathbf{A}^2\mathbf{B}$, $\mathbf{AB}^2$, $\mathbf{A}^2\mathbf{B}^2$.
W. Noll (Los Angeles, Calif.).

Raman, C. V., and Viswanathan, K. S. The elastic behaviour of isotropic solids. Proc. Indian Acad. Sci. Sect. A. 42 (1955), 1–9.

The authors attack the linear theory of elasticity because it requires the stress tensor to be symmetric and because it does not allow the stress to depend on rotations as well as strains. They derive equations for isotropic materials, allowing the stress to depend on rotations, and calculate wave speeds.

It has long been recognized that it is mathematically possible for an asymmetric stress tensor to depend on rotations as well as strains and, as is discussed by Whittaker [A history of the theories of aether and electricity, v. I, Philosophical Library, New York, 1951; MR **14**, 1], such relations have been considered as constitutive equations for an elastic aether. As the reviewer sees it, the authors merely reaffirm that this is a mathematical possibility, though they imply that they do more.

When inertial and body forces vanish, the stress tensor will give rise to zero resultant force and moments for every material volume if and only if it be symmetric. If one accepts their notion, one must therefore be prepared to believe that, with some possible exceptions, the applied surface tractions required to hold an elastic body at rest in a deformed position will not be in static equilibrium, or that they are, but there is some other mechanism such as an electric field whereby the resultant force and moment associated with the stress tensor can be balanced. The authors do not discuss this point, though a discussion is given in Whittaker. By reasoning which is not clear to the reviewer, the authors conclude from the fact that a rod may be twisted by couples applied to its ends, that the stress tensor should not be symmetric. Ordinary linear elasticity can, of course, provide a solution for such problems.

There is no mathematical objection to their proposal, which is not original with them. They present no concrete evidence that linear elasticity is inadequate to describe ordinary elastic solids or that there is a real material to which their equations apply. Consequently, some of their statements, such as "The neglect of rotations in the analysis of strains and of torques in the analysis of stress characteristic of that theory [linear elasticity] is shown to be unjustifiable," are difficult to accept.
J. L. Ericksen (Washington, D.C.).

Kączkowski, Zbigniew. The conjugate directions in an anisotropic body. Arch. Mech. Stos. 7 (1955), 52–86. (Polish. Russian and English summaries)

The author considers a homogeneous anisotropic elastic body and determines the 21 material constants a_{ik}'' of the stress-strain relations with respect to a rectilinear oblique system of coordinates and the corresponding material constants a_{ik} with respect to a Cartesian orthogonal system of coordinates. The relations obtained in this way between a_{ik}'' and a_{ik} contain also the nine cosines of the angles that determine the orientation of the oblique coordinate system with respect to the orthogonal one. The author then states that, since there are six mutually independent cosines out of the nine mentioned above, it is possible by a suitable choice of the oblique coordinate system to make six of the material constants a_{ik}'' vanish, so that the corresponding stress-strain relations will contain only 15 of them. If the structure of the body is such that it has with respect to the primary orthogonal system of coordinates only 15 constants, then according to the above statement the number of the independent material constants in the stress-strain relations can be reduced to nine. Such bodies the author calls paratropic and the directions of the corresponding oblique system of coordinates are conjugate elastic directions for a paratropic body. *T. P. Anđelić.*

Aymerich, Giuseppe. Una proprietà dell'energia elastica. Boll. Un. Mat. Ital. (3) 10 (1955), 332–336.

Let C be an elastic body held in static equilibrium by tractions applied to a part σ of its boundary, where the displacement vector is s. The remainder is assumed free. Conceptually one forms a larger body C' by adding material to C in such a way that σ remains on the boundary of C'. Let C' be deformed in any manner consistent with the condition that the displacement be s on σ. The author shows that, according to linear elasticity, the strain energy of C never exceeds that of C'. He indicates how this and previously known results can be used to obtain upper and lower bounds on the strain energy of solids.
J. L. Ericksen (Washington, D.C.).

Reiner, M. The complete elasticity law for some metals according to Poynting's observations. Appl. Sci. Res. A. 5 (1955), 281–295.

After pointing out a fallacy in an argument of Timoshenko regarding finite torsion, the author claims to

replace previous theoretical treatments of finite torsion by solving the problem in closed form. He then fits the result to some measurements of Poynting [Proc. Roy. Soc. London. Ser. A. **82** (1909), 546–559; **86** (1912), 534–561] on the extension of steel and copper wires in torsion. He concludes that Poynting's experiments are satisfactorily explained by a certain particular theory of finite elastic strain.

In the reviewer's opinion, no conclusion whatever is justified by the material in this paper. As the author has pointed out in many previous papers, his basic equations

$$p_j^i=F_0\delta_j^i+F_1e_j^i+F_2e_k^ie_j^k, \tag{1.3}$$

are invariant with respect to change of strain measure. In this paper, however, he does not use these equations, but rather he selects a particular strain measure and then takes F_1 and F_2 as constants. Thus he uses a particular quadratic theory, and in fact his results are less general than those of Murnaghan [Finite deformation of an elastic solid, Wiley, New York, 1951; MR **13**, 600], who uses the quadratic theory which in a certain sense is the most general. The author's criticism of this and other more general treatments is unjustified. Second, the author accepts Poynting's conclusion that the longitudinal extension is proportional to the square of the twist. It is well known that this result, with arbitrary ratio of extension to the square of the twist, can be fitted by any quadratic theory of elasticity with a disposable modulus, providing one makes "suitable approximations" such as those made by the author on his pp. 290–291.

C. A. Truesdell (Bloomington, Ind.).

Položiĭ, G. N. Variational-topological theorems on boundary problems of the theory of torsion of shafts of variable cross-section. The method of preservation of domain and of majorizing domains. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 245–270. (Russian)

Let $ABCDA$ be a contour bounding a region G in which AB, DC are equipotential curves ϕ=const. and AD, BC are stream lines ψ=const. If A is displaced to a new position A' on the curve AB, a new region G_1 bounded by the contour $A'BCDA'$ is formed. Other regions G_1 can be obtained from G by displacing various combinations of the original points in a similar fashion. By mapping G and G_1 upon their respective hodograph planes and using results of his earlier work [Dokl. Akad. Nauk SSSR (N.S.) **58** (1947), 1275–1278; Mat. Sb. N.S. **32(74)** (1953), 485–492; MR **9**, 507; **15**, 320], the author proves a sequence of theorems on the behavior of certain functions of ϕ and ψ as G is deformed into G_1.

The results are couched in the language of the problem of torsion of a shaft of revolution with variable cross-section. In this problem ψ is the stress function. As an application, numerical bounds for the maximum value of the stress vector are obtained for shafts with annular grooves of hyperbolic and circular section. The author claims that in particular cases his method furnishes a mathematical substantiation of St.-Venant's principle and gives to it a quantitative character. *R. N. Goss.*

Chien, Wei-Zang. Assumptions in Saint-Venant's solution for the torsion of an elastic cylinder. Chinese J. Phys. **9** (1953), 215–220. (Chinese. English summary)

In this paper, it is proved that Saint-Venant's solution for torsion can be obtained from the following simplified assumptions: $\partial\tau_{xz}/\partial z=\partial\tau_{yz}/\partial z=0$. These assumptions are much more simplified than those given by Saint-Venant in 1855, A. Clebsch in 1862, W. Voigt in 1887, and J. N. Goodier in 1937. *Author's summary* (Zbl **52**, 418).

Reissner, E. On torsion with variable twist. Österreich. Ing.-Arch. **9** (1955), 218–224.

This paper deals with the problem of Saint-Venant torsion of a cylindrical bar, one end-section of which is fully restrained against warping. The treatment constitutes an application of an important variational theorem due to the author [J. Math. Phys. **29** (1950), 90–95; MR **12**, 301], in a generalized form due to B. Fraeys de Veubeke [Bull. Service Tech. Aero. no. **24** (1951)]. The restriction to doubly symmetric cross-sections, made in an earlier paper on this subject [J. Math. Phys. **31** (1952), 214–221; MR **14**, 335], is abandoned in the present investigation.

The variational principle employed here is equivalent to the stress equations of equilibrium and to the displacement-stress relations of the linear theory of elasticity; it thus permits independent assumptions regarding the structure of the approximating stress and displacement fields. The particular assumptions made by the author are guided by Saint-Venant's solution for the case of unrestrained warping, but admit a variable angle of twist as well as a non-vanishing axial normal stress. By variational means the author then deduces ordinary differential equations and boundary conditions appropriate to the initially arbitrary elements in the assumed displacements and stresses. Without explicit integration of these equations, certain general qualitative conclusions are drawn as to the actual torque-twist relation and as to the location of the center of twist when warping of one end-section is prevented. *E. Sternberg* (Chicago, Ill.).

Gorgidze, A. Ya. Stretching and twisting of composite beams near to prismatic ones. Soobšč. Akad. Nauk Gruzin. SSR. **8** (1947), 605–612. (Russian)

A slightly tapered composite beam with one fixed end is subjected to the action of a (a) tensile force on the free end, and (b) torsion couple on the free end. If the lateral surface of the beam is determined by the equation $f[x(1-kz), y(1-kz)]=0$, where z is directed along the length of the beam and $kz\ll 1$, then the displacements in the problem (a) are given by formulas (17) and those for problem (b) by (27), provided that the Poisson constants of materials composing the beam are equal.

I. S. Sokolnikoff (Los Angeles, Calif.).

Gorgidze, A. Ya. Torsion and bending of composite beams near to prismatic ones. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 117–141 (1948). (Georgian. Russian summary)

In this paper a solution is given of the problems of torsion and bending for composite nearly prismatic bodies. By a composite body is meant a body composed of various materials in the following way. The undeformed beam consists of a series of parallel continuous rods, not touching one another, surrounded by an elastic medium filling the space between the bars and bounded on the outside by a cylindrical surface whose generators are parallel to the rods. By a nearly prismatic body is understood a body bounded by the surface

$$f[x(1-kz), y(1-kz)]=0,$$

where k is a small parameter, and by two planes parallel to the (x, y)-plane. *From the author's summary.*

Ruhadze, A. K. Bending by a normal force of a strongly twisted bar. Soobšč. Akad. Nauk Gruzin. SSR. 8 (1947), 291–298. (Russian)

The author studies the interaction of large torsional and bending deformations in a prismatic elastic rod, one end of which is fixed and the other is subjected to the action of a transverse force and twisting couple. The stress-strain relations are taken in the form proposed by Murnaghan. The displacements are assumed in the form of the familiar torsional and flexural displacements in linear elasticity plus the unknown displacements accounting for the effect of interaction. The equilibrium and compatibility equations then lead to the determination of 5 functions satisfying systems of equations (13) and (14) on pp. 296–297 of the paper. *I. S. Sokolnikoff.*

Gorgidze, A. Ya. Torsion of a stretched prismatic beam composed of various materials. Soobšč. Akad. Nauk Gruzin. SSR. 9 (1948), 161–165. (Russian)

A procedure analogous to that used in the paper reviewed above is followed to study the effects of torsion of a stretched composite elastic beam of uniform cross sections. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Gorgidze, A. Ya. The bending by a pair of forces of a stretched prismatic beam composed of various materials. Soobšč. Akad. Nauk Gruzin. SSR. 9 (1948), 539–545. (Russian)

A procedure analogous to that used in the paper reviewed second above is followed to study the effects of pure bending of a stretched composite elastic beam of uniform cross sections. *I. S. Sokolnikoff.*

Fraeijs de Veubeke, B. Aspects cinématique et énergétique de la flexion sans torsion. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° 29 (1955), no. 2, 48 pp.

This paper contains a lucid exposition of the theory of Saint-Venant torsion and flexure for cylinders with multiply connected cross-sections. Particular emphasis is placed on a review of various existing definitions of the center of twist (in the presence of restrained warping) and of the center of flexure. The author's own contribution consists of a kinematic interpretation of Trefftz's definition of the center of flexure. Let the z-axis be parallel to the generators of the cylinder, let D be its (multiply connected) cross-section, and D^* be the simply connected region bounded by the outer boundary of D. It is shown that the vanishing of a weighted average twist, expressed by

$$\iint_{D^*}\varphi^*\frac{\partial\omega_z}{\partial z}dxdy=0$$

is equivalent to the requirement of zero energy of interference between torsion and flexure, which underlies Trefftz's definition. Here φ^* in D coincides with the warping function for D, is constant in D^*-D, and continuous in D^*; ω_z is the analytic extension over D^* of the axial rotation component. *E. Sternberg.*

Colombo, G. Limitazioni superiori per i moduli delle componenti di stress in un particolare problema di deformazione piana. Ann. Univ. Ferrara. Sez. VII. (N.S.) 3 (1954), 45–54.

Using the explicit formulae of Gross [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6 (1949), 444–448; MR 11, 626] for the general solution of the problem of plane stress in linear elasticity with given stresses on a circular boundary, the author obtains simple upper bounds for the interior stresses in terms of the maxima of the boundary stresses. *C. A. Truesdell.*

Paria, Gunadhar. Stress distribution in thin aeolotropic plates. II. Bull. Calcutta Math. Soc. 46 (1954), 153–161.

[For part I see same Bull. 46 (1954), 103–107; MR 16, 769.] In a previous paper a method of Fourier transform of the stress function has been given for the solution of the generalised plane stress problems, when the material of the plate has two planes of elastic symmetry at right angles to each other and also at right angles to the plane of the plate. In the present paper, this method has been applied to find the stress distribution in (i) a semi-infinite plate with concentrated force acting at an internal point and (ii) a strip of infinite length but of finite breadth acted on by (a) a concentrated force and (b) an isolated couple within it. (Author's Summary.) *R. M. Morris.*

Melan, E. Spannungen infolge nicht stationärer Temperaturfelder. Österreich. Ing.-Arch. 9 (1955), 171–175.

Paper discusses a problem in which thermal stresses are induced by a non-steady temperature field in a body. A thin disk of infinite extent is initially stress-free and at a uniform temperature. Suddenly a quantity of heat is supplied at a point of the disk. The complete analytical solution, together with graphs of the temperature and radial and hoop stress distributions at various times, are found for this problem on the basis that the material is isotropic and elastic and that thin-plate theory is applicable. The radial displacement is not explicitly discussed. Reviewer's note: Extensions in which account was taken of a) application of heat over a finite area and b) plastic deformations would be valuable. *H. G. Hopkins.*

Oseled'ko, A. I. On bending of trapezoidal plates. Inžen. Sb. 21 (1955), 142–150. (Russian)

A thin elastic plate, trapezoidal in shape and built-in at its edges, is in equilibrium under uniform transverse pressure. Two approximate solutions, one based upon an energy method and the other upon Galerkin's method, are given of this problem. Some experimental results for the deflection of a plate are shown to be in very good agreement with theoretical results given by the energy method. *H. G. Hopkins* (Sevenoaks).

Tschech, E. Bemerkungen zur Theorie der Membranschalen. Österreich. Ing.-Arch. 9 (1955), 224–230.

This paper is a contribution to the membrane theory of thin elastic shells. Attention is confined to the case of shells in equilibrium under constant internal pressure. The three equilibrium equations are referred to orthogonal curvilinear co-ordinates taken along the principal curvature directions of the shell middle-surface. The Codazzi equations for the middle surface are used to rewrite these equations in different form. Some specialisation is made to spherical shells and shells of revolution. Reviewer's note: A tensor treatment of this topic is given by A. E. Green and W. Zerna [Theoretical elasticity, Oxford, 1954, Ch. XII; MR 16, 306]. *H. G. Hopkins* (Sevenoaks).

Ševlyakov, Yu. A. On integration of the equations of sloping spherical shells. Dopovidi Akad. Nauk Ukrain. RSR 1955, 235–237. (Ukrainian. Russian summary)

The author proves that in the general case the equilibrium equation for a spherical shell can be expressed as a

sum of the equilibrium equation for a plate on an elastic foundation and a certain harmonic equation. He derives in a closed form a general solution for an axi-symmetric sloping shell which differs from other known solutions for this case. A numerical example illustrates the theory.
T. Leser (Aberdeen, Md.).

Kornecki, Aleksander. On the equations of the theory of thin elastic shells. Arch. Mech. Stos. **6** (1954), 593–600 (1955). (Polish. Russian and English summaries)
In this paper the method of "complex forces", developed by V. V. Novožilov [Theory of thin shells, Moscow, 1951] for thin shells with respect to the principal lines of curvature, is generalized to shells referred to any orthogonal system of coordinates. The author derives a system of three partial differential equations with three unknown complex functions. These equations contain as special cases: the equations of Novožilov, those of the so-called "engineer's practical theory" of shells of V. Z. Vlasov [General theory of shells, Moscow-Leningrad, 1949; MR **11**, 627], and finally the equations of membrane theory.
T. P. Anđelić (Belgrade).

Mossakovskiĭ, V. I. The first fundamental problem of the theory of elasticity for a space with plane circular slit. Prikl. Mat. Meh. **19** (1955), 443–452. (Russian)
The author considers an elastic space with a plane circular slit inside. At the upper and lower walls of the slit the stresses are prescribed, and at the infinity all stresses vanish. There are four possible cases of the prescribed stress state at the upper and lower boundaries of the slit. The author considers only two of them, as the remaining two can be reduced to the cases solved by M. I. Leonov Prikl. Mat. Meh. **4** (1940), no. 5–6, 73–86; 17 (1953), 87–98; MR **2**, 365; **14**, 1146]. The author of this paper derives general solutions and illustrates his cases on examples where the prescribed stresses are constant. His solutions show that the results obtained by S. G. Mihlin [ibid. **10** (1946), 301–304] for a space weakened by one or more slits which are in the same plane are wrong.
T. Leser (Aberdeen, Md.).

Pekeris, C. L. The seismic surface pulse. Proc. Nat. Acad. Sci. U.S.A. **41**, 469–480 (1955).
The problem considered is that of the displacements of the surface of a semi-infinite elastic solid due to an internal point source. The source is a step-function in time, with a "splitting"-type compressive source-stress applied parallel to the normal to the surface at the source point. The results are restricted to equal Lamé constants.
Conventionally a problem of this sort is treated by employing a Fourier synthesis to express the time dependence of the source in terms of a harmonic time factor. The solutions of the wave equations are then expressed as Bessel contour integrals, and boundary conditions are applied to the computed stresses. A set of algebraic equations results whose solution gives the unknown integrands. The desired field quantities can then be determined from Fourier integrals of the results. These canbetransformed touseful form by deforming the contours.
The author modifies this procedure in that he employs Heaviside operational methods. He carries through the operator $p=\partial/\partial t$ instead of $p=i\omega$ (ω=circular frequency of the harmonic oscillation). Then (equations (24) and (51)) he employs integrals of the type $\int_0^\infty J_0(apx)f(x)dx$, where $f(x)$ is algebraic in x and $O(1/x)$ or $O(1/x^2)$ as $x\to\infty$. [What do they mean? For $p=i\omega$ they diverge.]
For a surface source the author completes his surface-displacement calculations in closed form in terms of elliptic integrals. The results are most provocative, and it is true, constitute a great improvement over those of Lamb. The graphs given are physically reasonable.
E. Pinney (Berkeley, Calif.).

Kovacs, A. Un exemple d'application de la transformation de Laplace. Ann. Univ. Sarav. **3** (1954), 223–238.
The relation between strain $\varepsilon(t)$ and stress $\sigma(t)$ for a visco-elastic solid is written in the generalized form $\sum A_n\varepsilon^{(n)}(t)=\sum a_n\sigma^{(n)}(t)$, where $\varepsilon^{(n)}$ means nth derivative of ε with regard to t and the summation extends from 0 to a highest order N, on which no limitation is imposed. Application of the Laplace transform to a function $h(t)$ is written symbolically $\mathcal{L}[h(t)]=h(\varrho)$. Then the differential equation is transformed into $\sigma(\varrho)=R(\varrho)\varepsilon(\varrho)$, where $R(\varrho)=\sum A_n\varrho^n/a_n\varrho^n$. From its analogy with Hooke's law this is called Hooke's operator-equation. The author shows how it can be used for discussion of viscoelastic behavior.
B. Gross (Rio de Janeiro).

Gol'denblat, I. V. The theory of small elastic-plastic deformations of anisotropic media. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 2, 60–67. (Russian)
This paper first develops a finite-strain theory of the small elastic-plastic deformations of isotropic media. The yield condition is assumed to involve the first two invariants of the stress tensor. This theory is then extended to the case of anisotropic media. *H. G. Hopkins.*

Gol'denblat, I. I. On the theory of small elastic-plastic deformations of anistropic media. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 619–622. (Russian)
This paper re-presents theory developed in the paper reviewed above. *H. G. Hopkins* (Sevenoaks).

Petriščev, P. P. Elastic-plastic strains in an anisotropic body. Vestnik Moskov. Univ. **7** (1952), no. 8, 63-72. (Russian)
The fundamental equations of a finite-strain theory of the elastic-plastic deformations of anisotropic material are first stated. These equations are specialised to the case of transverse isotropy, and are then appropriate to the discussion of deformations in rolled sheet. The theory is applied to the study of stress-strain curves obtained from tensile tests made on specimens cut from steel sheet at various angles to the rolling direction. The theoretical stress-strain curves appear to match the experimental results quite well. In some cases good agreement extends up to strains of about 8%.
H. G. Hopkins (Sevenoaks).

Černyak, M. I. An analytic expression of the volume strain under stretching in the elastic-plastic region. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 43–45. (Ukrainian. Russian summary)
A very brief discussion is given of elastic-plastic compressibility under conditions of uni-axial stress. The basis for the results stems from previous work by the author and seems to be at least partly empirical.
H. G. Hopkins (Sevenoaks).

Lomakin, V. A. Large strains of a tube and of a hollow sphere. Inžen. Sb. **21** (1955), 61–73. (Russian)
Analysis based upon finite-strain theory is given of the following problems: large plastic deformations of (a) thick-

walled circular cylindrical tubes (arbitrary constant axial strain assumed) and spherical shells under internal and external pressure and (b) thin-walled circular cylindrical tubes and spherical shells under external pressure.

Reviewer's note. Similar analysis for cylindrical tubes is due to C. W. MacGregor, L. F. Coffin and J. C. Fisher [J. Appl. Phys. **19** (1948), 291–297; MR **9**, 482]. These problems have also been very extensively treated on the correct basis of incremental theory. [See R. Hill, The mathematical theory of plasticity, Oxford, 1950, Ch. V; MR **12**, 303; W. T. Koiter, Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, Stam, Haarlem, 1953, pp. 232–251; MR **14**, 1148.] It is known in the case of cylindrical shells under internal pressure [C. W. MacGregor et al., loc. cit.], and it must also be true for spherical shells, that severe anisotropy develops during cold-work. More complete theories should therefore take account of the development and variation of anisotropy. *H. G. Hopkins* (Sevenoaks).

Mikeladze, M. Š. On the plastic state of a rotating anisotropic cylinder. Prikl. Mat. Meh. **19** (1955), 504–506. (Russian)

A circular cylinder is in steady rotation about its axis. The material is supposed anisotropic with rotational symmetry about the axis. This paper discusses stress distributions when the material is a) plastic-rigid and b) elastic.

Reviewer's note. The purpose of the investigation is to determine the bound set to the angular velocity by plastic deformation. The author appears to assume that plastic flow first occurs at the axis. This situation cannot be true generally because in the isotropic case plastic flow first occurs at the periphery. *H. G. Hopkins.*

Hodge, P. G., Jr. Impact pressure loading of rigid-plastic cylindrical shells. J. Mech. Phys. Solids **3** (1955), 176–188.

The paper is concerned with the dynamics of rigid-plastic cylindrical shells. The basic equations of the dynamic problem are established by adding inertia terms to the equations of equilibrium and by simplifying the yield curve of Drucker [Proc. 1st Midwestern Conference Solid Mech., Univ. of Illinois, 1953, pp. 158–163] by a square. The resulting equations are then solved for a typical bay of shell subjected to uniform external pressure which is brought on suddenly and after a certain time interval removed suddenly. It is found that four different types of solutions exist, depending upon whether the load is "medium" or "high" and the shell is "short" or "long". Expressions for the stresses, velocities, and displacements are found in each case, and typical results are presented graphically. *E. T. Onat* (Ankara).

See also: Amerio et al., p. 153; Duschek und Hochrainer, p. 214.

MATHEMATICAL PHYSICS

Have

★**Duschek, Adalbert, und Hochrainer, August. Grundzüge der Tensorrechnung in analytischer Darstellung. III. Teil: Anwendungen in Physik und Technik.** Springer-Verlag, Wien, 1955. vi+250 pp. DM 24.00; $5.70.

This is the third part of the authors' text on tensor analysis [for part I (1949) and II (1950) see MR **11**, 237; **12**, 397]. The following applications of tensor analysis are discussed: (1) Mechanics of a particle, of a system of particles, of a rigid body; (2) Mechanics of continuous media (elasticity and hydrodynamics); (3) Electromagnetic theory and related topics; (4) Theory of relativity.

In the discussion of mechanics, the authors consider the kinematics of the motion of a particle (velocity and acceleration along the tangent and normal directions, velocity and acceleration in moving reference frames), dynamics of a mass point and of a system of points, moment of a force, moment of momentum, center of mass and the laws of motion of a system of particles. The kinematics of rigid bodies and the properties of the inertia tensor furnish an introduction to the dynamics of rigid bodies. The laws of motion of rigid bodies are obtained by replacing summation over a system of mass points by integration. It should be noted that this transition calls for some care since differentiation of finite sums is equivalent to forming the sum of the derivatives when the derivative of each term exists, but additional conditions or other procedures are needed for infinite sums and integrals. Finally, the authors apply the theory to the spinning top and an elastic motion of a rigid body. The latter leads to an interesting eigenvalue problem.

The mechanics of continuous media is concerned with: finite elastic deformation theory; hydrodynamics of perfect and viscous fluids. Deformation theory is treated by expressing all tensors in terms of a Cartesian orthogonal coordinate system attached to the undeformed body. This leads to a simple and interesting treatment of the strain tensor including the compatibility conditions for infinitesimal deformations. However, a disadvantage results in treating the stress tensor. That is, either the stress tensor must be expressed in terms of the coordinates of the undeformed body, or infinitesimal displacements must be assumed. The fact that the stress tensor is a function of position only, the symmetry of the stress tensor, the generalized Hooke's law, the equilibrium relations are discussed. The subject matter of hydrodynamics is: kinematics of fluids (the so-called Euler and Lagrange viewpoints and the equation of continuity), the equations of motion in terms of the stress tensor and in the Navier-Stokes form.

The most extensive application of tensor analysis in this book is to electromagnetic theory. First, the authors develop the theory of double vector fields, A_j, B_j, where $B_j=\lambda A_j$ (λ is a positive scalar field). Three cases are discussed in detail: (1) A_j is irrotational and B_j has zero divergence; (2) A_j is irrotational; (3) B_j has zero divergence. The representation of these three fields by integrals furnishes many interesting analogies to theorems in potential theory. Following this theory, the authors consider the problem of heat conduction and the energy relation involving heat conduction, compressibility and viscosity. This last relation is very general and is determined by the energy-balance method. Electrostatic, magnetic, electric, and electromagnetic fields are treated in this order. The scalar and vector potentials of the electrostatic and magnetic fields, respectively, were discussed earlier under double vector fields (cases 2, 3). A great deal of the present theory deals with the energy of these fields. Further, the electric field due to moving charge is discussed and the energy of this field is determined. Finally, the Maxwell equations for the electro-

magnetic field are stated and applied to a study of the Poynting vector, and then electromagnetic fields which vary with time are discussed. In this latter topic, the retarded potential is analyzed for spherically and cylindrically symmetric waves.

The final section of the text is concerned with special relativity and general relativity. First, the authors compare the Galilean transformation of classical Newtonian physics with the Lorentz transformation of special relativity. This topic is followed by a discussion of the kinematics of special relativity (the decrease of length and increase in duration of events measured in a moving reference frame, the group property of the Lorentz transformations, the composition of velocities), the dynamics of special relativity (mass of a moving particle, force on such a particle, its energy, and the energy-impulse tensor of a continuous medium), and relativistic electrodynamics. Finally, the authors consider the general theory of relativity. Here, the Einstein field equations, the Schwarzschild line element (with applications to the advance of Mercury's perhelion and the bending of a light ray in the vicinity of the sun), the spherical world of DeSitter and the cylindrical world of Einstein are discussed.

This text is clearly written and the topics are discussed in orderly fashion. From the viewpoint of students, its only shortcoming is the absence of problems. On the other hand, because of the wide range of topics covered and the many interesting discussions, this text should serve as a good introduction to the uses of tensor analysis in physics and applied mathematics. Both the student and the specialist will find in this text many interesting items. *N. Coburn* (Ann Arbor, Mich.).

Whitrow, G. J. Why physical space has three dimensions. British J. Philos. Sci. **6** (1955), 13–31.

Optics, Electromagnetic Theory, Circuits

Lo, Y. T. Electromagnetic field of a dipole source above a grounded dielectric slab. J. Appl. Phys. **25** (1954), 733–740.

The paper presents a complete theoretical solution of the problem of a vertical electric dipole above a dielectric coated conducting plane. The solution is limited to the case of a perfect conductor and a perfect dielectric. The integral transformations involved are evaluated for the distant field by the saddle point method and by an extension of the method due to van der Waerden [Appl. Sci. Res. B. **2** (1951), 33–45; MR **12**, 808]. It is shown that for any finite dielectric thickness there exists at least one cylindrical surface wave in the sense of Sommerfeld [Ann. Physik (4) **28** (1909), 665–736]. The zero-order spherical wave has a null value at the dielectric surface except for a certain critical dielectric thickness which corresponds to the analytical case where the saddle point and a pole coalesce as the pole vanishes. For this condition a higher order solution must be considered.

W. K. Saunders (Washington, D.C.).

Brick, D. B. The radiation of a Hertzian dipole over a coated conductor. Proc. Inst. Elec. Engrs. C. **102** (1955), 104–121.

This paper, which was evidentally written about the same time as was the paper reviewed above, develops by similar methods the theoretical solution for the electric dipole above a dielectric coated conductor. It computes, as well, the solution for four additional cases; the electric dipole in the dielectric layer, the half electric dipole affixed to the plate, the magnetic dipole above the dielectric, and the magnetic dipole in the dielectric. The author's treatment, which was motivated by effects noted during measurements with oxidized metal sheets, does not include the critical case discussed in the paper reviewed above. Field patterns and the radiation resistance are given for all cases. *W. K. Saunders.*

Drăganu, Mircea. Essai d'une théorie approximative de la diffraction de la lumière par des corps matériels imparfaitement conducteurs. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 481–505 (1952). (Romanian. Russian and French summaries)

Kirchhoff's formulation of Huyghen's principle for perfectly absorbing bodies has been modified in order to take into account the finite conductivity of the diffracting obstacle. This modification is based on the assumption that in Green's formula the normal derivative should be replaced by a directional derivative making a finite angle with respect to the normal of the surface of the diffracting body (the electric field being no longer normal for real materials). This extension of Green's formula has been given by Picone and Miranda [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **29** (1939), 160–165] and also by Tolotti [ibid. **29** (1939), 285–293] and also by the author [Acad. Repub. Pop. Române. Bull. Şti. Ser. Mat. Fiz. Chim. **2** (1950), 567–574; MR **14**, 175]. By introducing the asymptotic expression for the electric field derived by Wise [Bell System Tech. J. **8** (1929), 662–671], for the case of reflection of electromagnetic waves by plane surfaces, into the modified formula, the author obtains an expression for the total excitation field which contains the Fresnel reflection coefficient, thus explaining the phenomena of polarization and selective reflection which have been observed by Gouy seventy years ago. *N. Chako.*

Vaĭnšteĭn, L. A. Diaphragms in wave guides. Ž. Tehn. Fiz. **25** (1955), 841–846. (Russian)

Vaĭnšteĭn, L. A. Diffraction of electromagnetic waves by a grid of parallel conducting strips. Ž. Tehn. Fiz. **25** (1955), 847–852. (Russian)

The two papers deal with mathematically identical problems. The method used, the Wiener-Hopf method, is the same as that used previously in the solution of the same problem by G. L. Baldwin and A. E. Heins [Math. Scand. **2** (1954), 103–118; MR **16**, 98]. *J. Shmoys.*

Derfler, Heinrich. On the theory of disc-loaded waveguide. Z. Angew. Math. Phys. **6** (1955), 190–206.

This paper is a sequel to a preceding paper [same Z. **6** (1955), 104–114; MR **16**, 978]. Here the methods developed there are applied to the theory of a disc-loaded waveguide. First the admittance-function is evaluated by introducing an approximation, that leads to Bessel and Hankel functions of orders 0 and 1, instead of functions of order 1/6, since there are no tables available for the latter.

The admittance-function is written in the form

$$G(\vartheta)=\frac{x}{\zeta}\left\{\frac{\pi\alpha}{\vartheta}R(x)-S(x,z)\right\},$$

where

$$R(x)=\frac{1}{x}\,\frac{J_1(x)Y_0(xba^{-1})-J_0(xba^{-1})Y_1(x)}{J_0(x)Y_0(xba^{-1})-J_0(xba^{-1})Y_0(x)}$$

and $S(x, z)$ is given by infinite series, whose terms are also expressed in Bessel functions. The series for $S(x, z)$ which are only slowly convergent are transformed into a form, more convenient for computation, by Kummer's transformation and the Mellin-inversion formulae. The concept of coupling-impedance is also discussed. The last § gives tables for the functions $R(x)$ and $S(x, z)$.

H. Bremekamp (Delft).

Poincelot, Paul. Réflexion d'une onde électromagnétique plane sur un gaz ionisé suivant une certaine loi. C. R. Acad. Sci. Paris **241** (1955), 290–292.

The author treats the propagation of a plane electromagnetic wave in a medium with refractive index given by the law $n=1$ $(x<0)$ and $n=(1-\alpha x/\omega^2)^{\frac{1}{2}}$ $(x>0)$, where x is the height and α and ω are constants. The field components satisfy a differential equation, solutions of which are Hankel functions of order one-third. The reflection coefficient is calculated under three assumptions concerning the magnitude of $\alpha h/\omega^2$ relative to unity, where h is the height at which the medium gives way to vacuum.

R. N. Goss (San Diego, Calif.).

Poincelot, Paul. Réflexion d'une onde électromagnétique plane sur un gaz ionisé. C. R. Acad. Sci. Paris **241** (1955), 649–651.

This note corrects the results of two previous papers [same C.R. **241** (1955), 186–188; and the paper reviewed above] in which a Hankel function had been replaced by an invalid asymptotic formula. In particular, a new asymptotic value for the reflection coefficient is obtained.

R. N. Goss (San Diego, Calif.).

Lampariello, Giovanni. L'equazione generale delle onde elettromagnetiche dei corpi in moto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 222–228.

Continuing previous work [same Rend. (8) **17** (1954), 37–44, 100–108; MR **16**, 775, 1076], the author is led to the following main result. In a medium moving with constant uniform velocity v, any component Φ of the electromagnetic vectors satisfies

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\Delta\Phi+\frac{n^2-1}{c^2-v^2}\frac{d^2\Phi}{dt^2}=0,$$

where $n^2=\varepsilon\mu$ and d/dt is the operation of differentiation following the medium.

J. L. Synge (Dublin).

Burštein, È., and Solov'ev, L. On the propagation of a principal wave between parallel surfaces. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 465–468. (Russian)

The paper deals with the propagation of electromagnetic waves between parallel curved surfaces in the dominant mode. It is shown that if the distance d between the surfaces is small, the local wave number differs from the free-space wave number by a term proportional to d^2. The coefficient of this term is calculated in terms of the curvatures of the surface. The deviation of rays from the geodesics of the curved surface is calculated in the first order.

J. Shmoys (Brooklyn, N.Y.).

Jancel, R., et Kahan, T. Théorie non maxwellienne des plasmas homogènes et anisotropes. Nuovo Cimento (9) **12** (1954), 573–612.

Den Ausgangspunkt der Untersuchungen der Verfasser bildet die im Zusammenhang mit der kinetischen Theorie der Gase hergeleitete und aus dieser Theorie bekannte Integro-Differentialgleichung von Boltzmann. In einem Plasma hat man zwei Verteilungsfunktionen f_1 und f_2, die sich auf die neutralen Atome bzw. auf die freien Elektronen beziehen. (Die Wirkungen der positiven Ionen kann man wegen deren grossen Massen vernachlässigen.) Unter der weiteren Annahme, dass auf das Plasma ein konstantes Magnetfeld und ausserdem ein oszillierendes elektrisches Feld einwirkt, erhalten dann die Verfasser die zwei Boltzmannschen Gleichungen

$$\frac{\partial f_1}{\partial t}+v_1\cdot \operatorname{grad}_r f_1=\frac{\partial e f_1}{\partial t},$$

$$\frac{\partial f_2}{\partial t}+v_2\cdot \operatorname{grad}_r f_2+F_2\cdot \operatorname{grad}_v f_2=\frac{\partial e f_2}{\partial t},$$

wo F_2 die Einwirkungen der äusseren Felder enthält und die rechten Seiten die infolge der Stösse auftretenden Änderungen der Funktionen f_1 und f_2 bedeuten. Da die Masse der Elektronen im Verhältnis zu der der Moleküle recht klein ist, so folgt für f_1 einfach eine Maxwellsche Verteilung; für f_2 ist das jedoch wegen dem Vorhandensein der äusseren Felder nicht der Fall. f_2 wird in nullter und erster Näherung berechnet und mit Hilfe dieser Resultate werden Formeln für die Leitfähigkeit, für den Tensor der Dielektrizitätskonstante, für den Halleffekt usw. hergeleitet, die in speziellen Fällen mit den bereits von anderen Autoren hergeleiteten Formeln übereinstimmen; jedoch mit dem Unterschiede, dass diese allgemeinen Berechnungen auch den Gültigkeitsbereich dieser Formeln einfach abzuschätzen gestatten. Dieselbe Bemerkung bezieht sich auch auf die im letzten Abschnitt bezüglich des Brechungsindexes, der Doppelbrechung, der Phasen- und Gruppengeschwindigkeit, der Absorption und der Polarisation in der Ionosphäre erhaltenen Resultate, wo man ebenfalls die bereits erhaltenen Formeln, jedoch mit deren genauem Gültigkeitsbereich zurückbekommt. Nur bezüglich der D-Schicht (wo die Stossfrequenz der Elektronen schon sehr gross ist) müssen die Rechnungen noch weiter verfeinert werden.

T. Neugebauer (Budapest).

Yakovlev, L. G. Velocity of the wave front in nonlinear electrodynamics. Ž. Eksper. Teoret. Fiz. **28** (1955), 246–248. (Russian)

Classical non-linear electrodynamics is defined by a Lagrange-function

$$L=L(K, I^2),\ K=\tfrac{1}{2}(E^2-H^2),\ I=(E\cdot H).$$

The author determines the propagation-velocity of a "signal" which consists of a small discontinuity in the first derivatives of the field-strengths, superimposed on constant fields E, H which are not necessarily weak. An explicit formula is given for the velocity, which in the general case is rather complicated. In the special case in which E, H and the direction of propagation are three orthogonal vectors, and L depends only on K, the velocity reduces to

$$v=[\alpha EH\pm(1+2\alpha K)^{\frac{1}{2}}]/[1+\alpha E^2],$$

$$\alpha=(d^2L/dK^2)/(dL/dK),$$

the ambiguous sign indicating the velocity of propagation

forward or backward in the given direction. For suitable choice of L, this velocity can exceed the velocity of light in vacuum. The author raises the question whether any general physical principle restricts the form of L so that propagation faster than light cannot occur.
F. J. Dyson (Princeton, N.J.).

Clauser, Emilio. Velocità della luce nei corpi isotropi in moto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 362–365 (1955).

The characteristics of Maxwell's equations in an isotropic medium at rest satisfy

$$\eta\mu(\tau_{/0})^2-\sum_{i=1}^{3}(\tau_{/i})^2=0,$$

where η, μ are constants and $\tau_{/i}=\partial\tau/\partial x^i$ ($i=0, 1, 2, 3$). By applying a Lorentz transformation, the author develops formulae for the velocity of propagation relative to a moving observer. He points out that in general this velocity is not obtained by compounding relativistically the velocity of the light relative to the medium and the velocity of the medium relative to the observer; this composition is valid rigorously only if the directions of the two velocities in question coincide. *J. L. Synge.*

Nicolau, Edmond. Au sujet de la relation de Lorentz. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5** (1953), 191–198. (Romanian. Russian and French summaries)

The purpose of this note is to discuss whether the Lorentz condition between scalar and vector potential

$$\text{(a)}\quad \operatorname{div} A+\varepsilon\mu\partial\varphi/\partial t=0 \quad\text{or}\quad \operatorname{div} A+\sigma\mu\varphi+\varepsilon\mu\partial\varphi/\partial t=0$$

follows from the field equations. The author has shown that for two cases, (i) perfect homogeneous and isotropic dielectrics, (ii) homogeneous and isotropic conductors, (a) is independent of the field equations. The proof runs briefly as follows. Writing the field equations and (1) in operator form in a cartesian system $\sum_{i=1}^{5} V_i P_{ij}=0$ ($P_{ii}=\square-(X_i)^2$, $X_i=\partial/\partial x_i$) and noticing that P_{ij}, $i\neq j$, contains X_j as factor, the author concludes that P_{ii} must also contain X_i as factor. Writing $U_i=X_iV_i^*$, the solutions of the transformed system $\sum_{i=1}^{5} V_i^*Q_{ij}=0$ yield relations between U_i^* ($i=1, 2, \cdots, 4$) only. This shows that U_5^* or condition (a) is independent of the field equations. *N. Chako* (New York, N.Y.).

★**Ollendorff, Franz. Elektronik des Einzelelektrons.** Springer-Verlag, Wien, 1955. xii+643 pp. $23.10.

There has long been a serious need for a book on the electronics of the free (i.e. unbound) electron that would emphasize the mathematical aspects of the subject while at the same time giving concrete illustrations and describing the experimental arrangements. Therefore the appearance of Ollendorff's text is a cause for jubilation for anyone interested in electron devices. Within its tremendous wealth of material the book encompasses thorough treatments not only of electron optics proper but also of the often neglected but highly important electron ballistics. A chapter is devoted to the relativistic mechanics of the electron with applications to the design of particle accelerators. The chapter on Gaussian dioptrics discusses in detail the potential fields formed by various electron lenses, which is another useful feature of the book. Concrete examples and analogies are given throughout to illustrate the formulas developed.

The mathematics, unfortunately, is unnecessarily ponderous. For example, many intermediate steps are given which are perfectly obvious to anyone with the mathematical background needed to read the book, thus giving superfluous bulk. It is to be hoped that this will be modified if an English edition is issued. Other suggestions for this second edition are: the collection of all symbols used in a given chapter to be placed at the beginning of that chapter (to eliminate guesswork when the text is used for occasional reference) and a greatly expanded bibliography. The book is printed beautifully. However, the question arises whether a less artistic presentation and display of the formulas (together with the above-mentioned reduction in their number) would not result in a reduction in printing costs and price of the book. Its present price makes it prohibitive for graduate students, who could really benefit by it.

The reviewer's chief criticism of the book is directed at Ollendorff's failure to give credit to the originators of many of the techniques described. This is particularly objectionable in a field as new as electronics, where most of the inventors are presumably still alive. To give but one example of this neglect: deflection focusing in a parallel plate condenser is discussed without mentioning the name of Deserno, who was the first to derive an explanation of the effect. *J. E. Rosenthal.*

Glaser, Walter. Eine neue Begründung der wellenmechanischen Elektronentheorie. Österreich. Ing.-Arch. **8** (1954), 110–120.

By introducing a magnetization of the vacuum the author has derived on the basis of the energy-momentum relation a generalized form of the Maxwell-Lorentz field equations. The displacement and magnetic field vectors are found to satisfy equations of Klein-Gordon type. By a proper combination of the field vectors the new field equations are transformed to the Dirac equations provided one identifies the parameter K_0 (ratio of the magnetization of the vacuum to the energy) appearing in the field equations with $mc/\hbar$. In this theory the electron is considered as a space field distribution of energy and momentum. The latter quantities also satisfy equations of the Klein-Gordon type. The charge, current and energy flux densities and also the angular and magnetic moment of the electron can be calculated from the solutions of the energy and momentum equations. In the special case of static fields, the author has derived expressions for the "electron radius" and its charge density distribution in space. The ratio of the mechanical to the magnetic moment apart from a factor 2 agrees with the classical result (spherical symmetric electron). Finally, the author gives solutions of the field equations in terms of plane waves. *N. Chako.*

Bogolyubov, N. N., and Zubarev, D. N. The method of asymptotic approximation for systems with revolving phase and its application to the motion of charged particles in a magnetic field. Ukrain. Mat. Ž. **7** (1955), 5–17. (Russian)

The physical problem attacked by the authors is that of describing the non-relativistic motion of a charged particle in a non-homogeneous magnetic and electric field under the assumption that the Larmor frequency $\omega_H=eH/mc$ is large and that the magnetic field does not change greatly over the Larmor circle, or, as expressed by the authors $R_LH^{-1}dH/dx\ll 1$, R_L being the radius of

the Larmor circle, $R_L = w/\omega_H$, where w is the component of velocity of the particle in a plane perpendicular to the magnetic field. [This particular physical problem was previously considered by H. Alfvén, Ark. Mat. Astr. Fys. **27A** (1941), no. 22; MR **6**, 168.] By means of an asymptotic expansion valid for large values of the frequency, the motion of the particle can be described as the superposition of the rotation of the particle on the Larmor circle and the motion of the center of gravity of the circle. The result of the authors can be described in these terms: Let $\boldsymbol{\epsilon}_0$, $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ be a moving orthonormal frame, where $\boldsymbol{\epsilon}_0$ is parallel to the magnetic field, $\mathbf{H}$, and $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ lie in a plane perpendicular to $\mathbf{H}$. Let $\mathbf{v}$ be the velocity of the particle and let u be the component of the velocity parallel to $\mathbf{H}$, let w be the magnitude of $\mathbf{v} - u\boldsymbol{\epsilon}_0$, and let $\mathbf{r} = \int \mathbf{v}\, dt$. Then

$$\mathbf{r} = \bar{\mathbf{r}} + \left(\frac{w}{\omega_H}\right)(\boldsymbol{\epsilon}_2 \cos \bar{\alpha} - \boldsymbol{\epsilon}_1 \sin \bar{\alpha}), \quad \alpha = \bar{\alpha} + O\left(\frac{1}{\omega_H}\right),$$

α being the angular coordinate, where furthermore

$$\frac{d\bar{\mathbf{r}}}{dt} \approx u\boldsymbol{\epsilon}_0 - \left\{\left(-\frac{1}{\omega_H}\right)\mathbf{F} + \left(\frac{w^2}{2\omega_H{}^2}\right)\nabla\omega_H + \left(\frac{u^2}{\omega_H}\right)(\boldsymbol{\epsilon}_0\nabla)\boldsymbol{\epsilon}_0\right\} \times \boldsymbol{\epsilon}_0,$$

$$\frac{dw}{dt} \approx -\left(\frac{uw}{2}\right) \operatorname{div} \boldsymbol{\epsilon}_0, \quad \frac{du}{dt} \approx \mathbf{F}\cdot\boldsymbol{\epsilon}_0 + \left(\frac{w^2}{2}\right) \operatorname{div} \boldsymbol{\epsilon}_0,$$

$\mathbf{F}$ being the force of the electric field. The authors further obtain the fact that $w^2/H =$ const. $+O(1/\omega_H)$, expressing the fact that the magnetic flux through the Larmor circle is nearly constant. They point out in conclusion that the same methods may be used in studying gyroscopic motion. *W. L. Baily* (Princeton, N.J.).

Moon, Parry, and Spencer, Domina Eberle. On the Ampère force. J. Franklin Inst. **260** (1955), 295–311.

Graffi, Dario. Su una legge di minimo della magnetostatica. Ann. Univ. Ferrara. Sez. VII. (N.S.) **3** (1954), 25–29.

This paper deals with the magnetic field $(\mathbf{H}, \mathbf{B})$ due to steady currents in the presence of magnetizable bodies, in general ferromagnetic and anisotropic (but without hysteresis). A general relationship $\mathbf{B} = \mathbf{B}(\mathbf{H})$ is assumed, subject to the following conditions: (i) $\mathbf{H}\cdot d\mathbf{B} = dp(\mathbf{H})$, an exact differential, and (ii) $[\mathbf{B}(\mathbf{H}+\mathbf{H}') - \mathbf{B}(\mathbf{H})]\cdot\mathbf{H}' > 0$ for $\mathbf{H}' \neq 0$. The function $p(\mathbf{H})$ is energy density, and the total complementary energy is $U = \int q(\mathbf{H}) dS$ integrated throughout all space, where $q(\mathbf{H}) = \mathbf{H}\cdot\mathbf{B} - p(\mathbf{H})$. Let $\mathbf{H}_0$ be the field due to the currents in the absence of magnetizable bodies. Then any magnetic field $\mathbf{H} = \mathbf{H}_0 -$ grad ϕ is regarded as compatible with the currents, ϕ being subject to certain conditions as to vanishing at infinity. The minimum principle established in the paper is this: in the class of all fields compatible with the currents, the actual field due to the currents in the presence of the magnetizable bodies minimizes the total complementary energy. *J. L. Synge* (Dublin).

Have

★ **Bayard, M. A note on the arithmetical specification of messages and the use of this concept for the determination of corrective networks.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 131–144. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

The author assigns a number (to some base) to a "message", plots graph of this number versus "distortion" and from this determines certain corrective networks. *C. Y. Lee* (Chatham, N.J.).

Mandelbrot, Benoît. Théorie de la pré-correction des erreurs de transmission (théorème de A. Feinstein). Ann. Télécommun. **10** (1955), 122–134.

See also: Duschek und Hochrainer, p. 214.

Quantum Mechanics

★ **Heisenberg, W. The development of the interpretation of the quantum theory.** Niels Bohr and the development of physics, pp. 12–29. McGraw-Hill Book Co., New York, N.Y., 1955. $4.50.

Bopp, Fritz. Stochastische Darstellung der Quantenmechanik. Naturwissenschaften **42** (1955), 531–532.

Halbwachs, Francis, Lochak, Georges, et Vigier, Jean-Pierre. Modèle de la théorie causale des micro-objets relativistes de spin quelconque au moyen d'un fluide relativiste doté de moment cinétique interne. C. R. Acad. Sci. Paris **241** (1955), 744–746.

Potier, Robert. Sur les produits scalaires de fonctions d'onde et les intégrales de Fourier réciproques en mécanique ondulatoire relativiste. J. Phys. Radium (8) **16** (1955), 688–692.

Linear relativistic wave equations for particles can be expressed in either four-momentum space as algebraic equations or in space-time as differential equations. For the simple cases of spin zero and spin one-half, covariant expressions for the scalar product in the two spaces have been known for a long time [for a clear mathematical exposition, see M. Riesz, C. R. Dixième Congrès Math. Scandinaves, Gjellerup, Copenhagen, 1946, pp. 123–148; MR **8**, 427]. In the present paper, the analogous relations and expressions are worked out for a general equation containing an arbitrary set of spins and non-zero masses assuming that there is a conserved bilinear form which satisfies a certain reality condition and vanishes for any pair of states of the same four-momentum but different spin. Expressions are given for the scalar product, the correspondence between momentum space and space-time representations, and the propagation function of the particle. *A. S. Wightman* (Princeton, N.J.).

Choquard, Ph. Traitement semi-classique des forces générales dans la représentation de Feynman. Helv. Phys. Acta **28** (1955), 89–157.

Feynman has shown [Rev. Mod. Phys. **20** (1948), 367–387; MR **10**, 224] that in the Schrödinger theory the kernel representing the transformation of the wave function over a finite time interval can be expressed as the functional integral over all trajectories with given end-points of a classical quantity, obtained from considering the classical action between two neighboring points. The present paper, forming the doctoral thesis of the author, is devoted to a detailed study of a few questions connected with this formula of Feynman. The main point discussed is the possibility, not considered by Feynman, that in calculating the classical action between two neighboring points one has besides the direct (very short) classical trajectory joining the points a number of indirect trajectories of non-infinitesimal length. The author establishes that the contribution of these indirect trajectories to the classical action is negligible in the formula of

Feynman. The derivation, presented in ch. III, is founded on a general analysis of indirect trajectories developped in ch. II. Ch. IV is devoted to the calculation of the Feynman kernel for a variety of systems: non-conservative systems, particles with spin, systems described in curvilinear coordinates and relativistic particles.

L. Van Hove (Utrecht).

Newman, Ezra, and Bergmann, Peter G. Lagrangians linear in the "velocities". Phys. Rev. (2) **99** (1955), 587–592.

The authors apply Dirac's methods [Canad. J. Math. 2 (1950), 129–148; 3 (1951), 1–23; MR **13**, 306] for dealing with theories for which the momenta canonically conjugate to the field variables are not all algebraically independent of the field variables and their spatial derivations, to theories arising from Lagrangians linear in the first time derivatives of the field variables. They classify the possibilities that may arise into three classes: (1) that class in which the field equations are incompatible; (2) the field equations lead to proper Cauchy-Kowalewski problems; and (3) the solutions contain arbitrary functions. Methods for quantizing the compatible cases are indicated. *A. H. Taub* (Urbana, Ill.).

Schrödinger, E. The wave equation for spin 1 in Hamiltonian form. Proc. Roy. Soc. London. Ser. A. **229** (1955), 39–43.

The Proca equations for a particle of spin 1 are written in such a form that their identity with the beta-matrix formalism of N. Kemmer [same Proc. **173** (1939), 91–116; MR **1**, 95] is immediately apparent. The equations so obtained have also the advantage of being in the form

$$i\, d\psi/dt = H\psi,$$

where H is a matrix Hamiltonian, that is to say they are ordinary Schrödinger equations. *F. J. Dyson.*

Schönberg, M. On the hydrodynamical model of the quantum mechanics. Nuovo Cimento (9) **12** (1954), 103–133.

The paper gives a new interpretation of the Schrödinger equation in terms of a hydrodynamical model. It differs from the model proposed long ago by Madelung [Z. Physik **40** (1926), 322–326] through the occurrence of fluctuations; the Madelung model is found again when averages are taken over these fluctuations. The latter are said to correspond to turbulence. The formal procedure used by the author is a second quantization applied to the Schrödinger theory. Non-relativistic particles with arbitrary value of the spin are considered. *L. Van Hove.*

Schönberg, M. A non-linear generalization of the Schrödinger and Dirac equations. Nuovo Cimento (9) **11** (1954), 674–682.

A formal generalization is proposed for the Hamilton-Jacobi equation of classical mechanics. The point is to replace the relation $p_i = \partial S/\partial x_i$ between momentum and action function by the more general relation

$$p_i = (\partial S/\partial x_i) + \lambda(\partial \mu/\partial x_i),$$

λ and μ being functions of the space coordinates x_i and of the time. A fluid with particles following each of the trajectories corresponding to such a relation between p_i and the space and time coordinates is then no longer bound to have an irrotational motion. Similar generalizations are carried out for the Schrödinger and Dirac equations. *L. Van Hove* (Utrecht).

Schönberg, M. Simple solution of the generalized Schrödinger equations. Nuovo Cimento (9) **12** (1954), 300–303.

In the paper reviewed above the author has proposed a generalization of the Schrödinger equation corresponding to the inclusion of vorticity in the associated Madelung fluid motion. The present letter-to-the-Editor gives an example of solution of the generalized equation.

L. Van Hove (Utrecht).

Fradkin, E. S. On quantum field theory. I. Ž. Eksper. Teoret. Fiz. **29** (1955), 121–134. (Russian)

This paper gives a derivation of results stated without proof in a previous note [same Ž. **26** (1954), 751–754; MR **16**, 317]: The functional equations of Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 530] are re-expressed in terms of renormalized quantities. In addition, the original unrenormalized equations are derived by a method which does not depend on Schwinger's variational principle. Instead, functional differential equations are obtained for an operator $S(\infty)$, which describes the effect of external sources over the time interval $-\infty$ to $+\infty$. Matrix elements of these equations for $S(\infty)$, evaluated for zero external source strengths, yield Schwinger's equations. Furthermore, expansions of the Green's functions in power series in the external sources are given [see B. L. Ioffe, Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 761–764; MR **16**, 100, and references quoted there for earlier work on this type of series]. An appendix contains an explicit determination of $S(\infty)$ for zero coupling strength, and a proof of Furry's theorem. *A. S. Wightman* (Princeton, N.J.).

Fradkin, E. S. On the problem of interaction of two quantum fields. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 897–900. (Russian)

Explicit formal solutions by means of functional integrals are obtained for the equations of the quantum field theory of nucleons and mesons with pseudoscalar interaction. *F. J. Dyson* (Princeton, N.J.).

Lehmann, H., Symanzik, K., und Zimmermann, W. Zur Formulierung quantisierter Feldtheorien. Nuovo Cimento (10) **1** (1955), 205–225.

The program followed in this paper is to deduce from general principles of field theory, without using local field equations, relations which the Green's functions and scattering amplitudes of any field theory must satisfy. Relations so deduced are automatically renormalized, that is to say they are relations between finite and in principle observable quantities, and only experimental masses and coupling-constants occur in them. The authors here succeed for the first time in deducing such relations in sufficient quantity, so that there is a chance they may be sufficient to determine the observable consequences of the theory. Whether the theory is determined in this sense remains an open question. If the answer is affirmative, the authors will have reached the long-desired goal of reformulating field theory in such a way that renormalization is avoided completely.

The methods are quite general but are here applied for simplicity to a single scalar field $A(x)$ which may be linear or non-linear. The basic mathematical device is the reduction formula

$$(1)\quad (\Omega, T(x_1, \cdots, x_n)\Phi_{\alpha}{}^{in}) = -i\int f_\alpha(y) K_y \tau(x_1, \cdots, x_n, y) d_4y.$$

The notations are as follows. Ω is the vacuum state. f_α

is a normalized solution of the Klein-Gordon equation

$$K_y f_\alpha(y)=0,\ K_y=(\partial/\partial y_\mu)^2-m^2. \tag{2}$$

$\Phi_{|\alpha}{}^{\alpha}$ is a state with just one incoming particle described asymptotically by the wave-function f_α. $T(x_1, \cdots, x_n)$ is the chronological product of the field-operators $A(x_1), \cdots, A(x_n)$. The Green's function τ is defined by

$$\tau(x_1, \cdots, x_n)=(\Omega, T(x_1, \cdots, x_n)\Omega). \tag{3}$$

The proof of (1) is very simple and depends only on the assumption of the asymptotic condition, which requires that $A(x)$ tends as the time $x_0\to\pm\infty$ to an operator describing free particles with the observed rest-mass m. The local properties of $A(x)$ are not here involved. Generalizations of (1) to more than one incoming or outgoing particle are similarly obtained. In consequence, all physically observable matrix elements, between states in which ingoing or outgoing particles are specified, are reduced to integrals containing only the vacuum Green's functions τ.

The next step is to use the completeness condition, which requires that the states specified by ingoing or outgoing particles form a complete set. This condition may be written

$$\sum_{(\alpha)} (\Phi_{|\alpha}{}^{(\alpha)})(\Phi_{|\alpha}{}^{(\alpha)})=1. \tag{4}$$

Using (4) we may express every τ-function as a sum of products, each product consisting of one matrix element (1) multiplied by the complex conjugate of another. Thus each τ is equated to an integral in which the same or different τ-functions occur bilinearly. These equations relating the τ-functions are the main result of the analysis, and the scattering equation of F. E. Low [Phys. Rev. (2) **97** (1955), 1392–1398] is a special case of them.

In the last section the analysis is carried further with the additional hypothesis that the τ-functions are expandable in powers of a coupling parameter g. In this case it is proved that the non-linear effects proportional to g are local in character, that is to say they can be derived from a local field theory with a point-interaction term proportional to g in the Lagrangian, although no hypothesis of local behavior was made in the proof. It thus appears likely that the relations between τ-functions are essentially sufficient to determine the complete dynamics of the system, so long as power-series expandability is assumed.

In an appendix these results are derived by an alternative method using functional integration.

F. J. Dyson (Princeton, N.J.).

Skyrme, T. H. R. Quantum field theory. Proc. Roy. Soc. London. Ser. A. **231** (1955), 321–335.

An extension of Feynman's method of quantization [Phys. Rev. (2) **76** (1949), 749–759] is proposed which starts with the definition of a functional analogous to the scattering matrix of quantum theory. The principle of quantization used is then the following: "The transformation function from initial to final state should be proportional to the 'average' value of $U=\exp(-iJ/\hbar)$, averaged over all motions or paths disregarding the classical equations of motion." Here J is the transformation function of the classical problem, and $\hbar$ is introduced as a "temperature", in the same way as T is introduced in the definition of the partition function in statistical mechanics. "The deeper physical meaning that may be contained in this formal expression" is not elucidated. Furthermore it is shown that the functional U affords a "natural" four-dimensional representation for quantum field theory. In the following sections, alternative formulae are given for U, showing that in the interaction representation it is a probability amplitude for transitions between states with some definite quanta present. Applications to perturbation theory and to the problem of the "particle propagator" are finally proposed, together with a variational principle, a relativistic analogue of the variational principle leading to the Schroedinger equation. Very dense formalism makes the paper rather difficult to read. *D. Rivier* (Lausanne).

Polkinghorne, J. C. On the Feynman principle. Proc. Roy. Soc. London. Ser. A. **230** (1955), 272–276.

The equivalence of the Feynman method of setting up a quantum field theory [Phys. Rev. (2) **84** (1951), 108–128; MR **13**, 410] with the usual canonical formalism is here proved. This is not a new result, but a clear statement and proof of it is hard to find elsewhere in the literature.

F. J. Dyson (Princeton, N.J.).

Stepanov, B. M. On the introduction of dynamical variables in quantum field theory. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 889–892. (Russian)

This is a purely formal discussion of the definition and transformation properties of dynamical variables in the "intermediate representation" of N. N. Bogolyubov [same Dokl. (N.S.) **82** (1952), 217–220; MR **13**, 711]. The formulae are too complicated to be usefully summarized, but the basic definition is the following. A dynamical variable $A(x|g)$ is a function of the space-time point x and a functional of the real function $g(y)$. It satisfies the functional differential equation

$$\delta A(x|g)/\delta g(y)=i[A(x|g),\ L(x|g)] \text{ for } y_0>x_0,$$

and

$$\delta A(x|g)/\delta g(y)=0 \text{ for } y_0<x_0.$$

Here $L(x|g)$ is the Lagrangian introduced by Bogolyubov.

F. J. Dyson (Princeton, N.J.).

Bell, J. S. Time reversal in field theory. Proc. Roy. Soc. London. Ser. A. **231** (1955), 479–495.

Invariance of a physical theory under time reversal is often considered as a criterion for the validity of this theory. There are mainly two ways of defining the transformation of time reversal: the first way, introduced in quantum mechanics by Kramers [Akad. Wetensch. Amsterdam Proc. **33** (1930), 959–972, p. 962] and Wigner [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1932**, 546–559] changes the time coordinate t to $-t$; the second way, introduced more recently by Schwinger [Phys. Rev. (2) **82** (1951), 914–927; MR **13**, 520] involves in addition the change of the charge e to $-e$. It is shown in this paper that in classical field theory invariance under time reversal in the Schwinger sense may be deduced from proper isochronous Lorentz invariance and from invariance under spaee inversion. Using throughout the canonical formalism, invariance under time reversal is then extended to quantized fields, namely Bose, Dirac and Majorana fields. The connection with Wigner time-reversal invariance is given and finally properties of transition-probability amplitudes are deduced from time-reversal invariance. It is claimed that time reversal in the Schwinger sense rather than in the Wigner one arises "naturally" from a field-theoretical approach. This point does not seem clear to the reviewer, in so far as what looks "natural" is not a matter of taste. *D. Rivier.*

Klepikov, N. P. Solution of the system of equations for a vacuum functional. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 1057–1059. (Russian)

The functional form of the equations of quantum electrodynamics [Klepikov, same Dokl. (N.S.) **98** (1954), 937–940; MR **16**, 887] is studied with a view to obtaining a solution in the form of a power series in $(1/e)$. It is stated without proof that no such solution exists, because the equations defining the zero-order and first-order terms of the expansion are inconsistent. The author then proceeds to write down a formal solution in terms of functional integrals, identical with that found independently by E. S. Fradkin [ibid. **98** (1954), 47–50; MR **16**, 317]. *F. J. Dyson* (Princeton, N.J.).

Güttinger, Werner. Konvergente Reihenentwicklungen und singuläre Potentiale in der Quantenfeldtheorie. I. Z. Naturf. **10a** (1955), 257–266.

This paper consists of two parts. The first is a detailed exposition of the elementary properties of pseudo-functions in the theory of distributions [L. Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833]. The second consists of some incomplete applications of this theory to physical problems. It is claimed that the method gives a definite meaning to the sums of divergent series expansions in several field-theoretical problems. Details are to be published in a subsequent paper. *F. J. Dyson* (Princeton, N.J.).

Hamilton, J. Functional analysis and strong-coupling theory. Phys. Rev. (2) **97** (1955), 1390–1391.

The functional method of S. F. Edwards and R. E. Peierls [Proc. Roy. Soc. London. Ser. A. **224** (1954), 24–33; MR **15**, 1010] is applied to the "strong-coupling" treatment of a nucleon interacting with a meson field. This paper is only a statement of a program, and no results are obtained. *F. J. Dyson* (Princeton, N.J.).

Gol'fand, Yu. A. Construction of propagation functions by the method of quasi-fields. Ž. Eksper. Teoret. Fiz. **28** (1955), 140–150. (Russian)

The formalism of quantum field-theory is reconstructed starting with the notion of a "quasi-field" which is an operator $\phi(x)$ satisfying the commutation law

$$[\phi(x), \phi(y)]=0$$

and having a vacuum expectation value given by

$$\langle\phi(x)\phi(y)\rangle=\tfrac{1}{2}\Delta_F(x-y),$$

where Δ_F is the Feynman propagation function. This method is identical with that found independently by F. Coester [Phys. Rev. (2) **95** (1954), 1318–1323; MR **16**, 320]. It leads to a theory which is precisely equivalent to ordinary quantum field theory.

The second half of the paper deduces from the "quasi-field" approach explicit solutions of the equations of the theory, expressed in terms of functional integrals. There is a close connection between the quasi-fields and the functions which appear as integration-variables in the functional integrals. *F. J. Dyson* (Princeton, N.J.).

Cini, M., and Fubini, S. Non perturbation treatment of scattering in quantum field theory. Nuovo Cimento (9) **11** (1954), 142–152.

The variation principle given by Lippmann and Schwinger for ordinary scattering problems [Phys. Rev. (2) **79** (1950), 469–480; MR **12**, 570] is used to formulate an approximation method for scattering problems in field theory, different from the perturbation and Tamm-Dancoff expansions. The nth order approximation formula proposed by the authors, although not equivalent with it, involves the perturbation expansion of the S-matrix up to order n, and renormalization must be carried out by the Dyson method in this perturbation expansion before its insertion in the new formula. The S-matrix obtained is unitary in each order of approximation. No difficulties occur from non-physical poles in the propagation functions. The method is well suited for practical calculations of two-body collisions. *L. Van Hove* (Utrecht).

★ **Berestezki, W. B. Die Störungstheorie in der Quantenelektrodynamik.** Abhandlungen aus der Sowjetischen Physik, Folge IV, pp. 11–53. Verlag Kultur und Fortschritt, Berlin, 1954.

Translation of Uspehi Fiz. Nauk **46** (1952), 231–278.

Archibald, W. J. Field equations from particle equations. Canad. J. Phys. **33** (1955), 565–572.

Kemmer's equation for the meson field [Proc. Roy. Soc. London. Ser. **173** (1939), 91–116; MR **1**, 95] are written in a more explicit form by using a special 10 by 10 representation for the β_μ (here denoted by α_μ). The connection with the usual Proca equations is given. A generalization of the Duffin-Kemmer scheme is mentioned and shown to lead to the same results.
N. G. van Kampen (Utrecht).

Barašenkov, V. S. Some remarks on possible formulations of the theory of extended particles. Ž. Eksper. Teoret. Fiz. **28** (1955), 579–583. (Russian)

This article consists largely of comments about various papers on non-localized field theory. *N. Rosen* (Haifa).

See also: Newton and Jost, p. 155.

Thermodynamics, Statistical Mechanics

de Groot, S. R. Termodinamica dei processi irreversibili e generalizzazioni delle relazioni di Onsager. Nuovo Cimento (9) **12** (1954), supplemento, 5–16.

The paper reviews some recent contributions to the thermodynamics of irreversible processes. The principles of the formalism are first recalled, then illustrated on the case of heat conduction in one dimension and in an anisotropic crystal. A difficulty in the usual application of Onsager's reciprocity relations is then mentioned and its solution is given by a method due to Casimir [Rev. Mod. Phys. **17** (1945), 343–350] as well as by a slightly different method. *L. Van Hove* (Utrecht).

Prigogine, I., et Bingen, R. Sur la mécanique statistique des phénomènes irréversibles. IV. Physica **21** (1955), 299–311.

This paper extends to three-dimensional crystals the results established previously by Klein and Prigogine [Physica **19** (1953), 1053–1071; MR **15**, 491] for the simplest type of one-dimensional crystal. Harmonic forces are assumed and quantum effects are neglected. The dependence of positions and velocities of all atoms at time t upon their values at time 0 is derived by the usual method of expansion in normal modes and its nature is discussed for large t. One then studies the statistical

distribution of positions and velocities for large t assuming for the initial conditions a statistical distribution without long range correlations. Gaussian distributions are obtained in the limit of long times, but of course not the equilibrium distribution, since anharmonic forces are neglected. *L. Van Hove* (Utrecht).

Bogolyubov, N. N. Equations with variational derivatives in problems of statistical physics and quantum field theory. Vestnik Moskov. Univ. **10** (1955), no. 4–5, 115–124. (Russian)

The text of a lecture surveying the field of variational equations in physics. The main thesis is that the method of variational differentiation, introduced into quantum field theory by J. Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 520], can be used equally well in classical statistical mechanics. Thus for a system of interacting particles with energy $U(r_1, r_2, \cdots, r_N)$, a fictitious external field $\varphi(r)$ may be introduced into the free energy

$$F=-\Theta \log \int \exp [-\Theta^{-1}(U+\sum_j \varphi(r_j))]dr_1\cdots dr_N.$$

Then the variational derivatives of F with respect to φ give all possible joint distribution functions of any finite number of particles. Equations for these distribution functions may be derived in this way very succinctly. *F. J. Dyson* (Princeton, N.J.).

Kümmel, H. Zur quantentheoretischen Begründung der klassischen Physik. I. Dynamik der Gase und Flüssigkeiten. Nuovo Cimento (10) **1** (1955), 1057–1078.

The aim of this paper is to deduce consistently the classical dynamical equations for gases and fluids from the quantum theory. Some obscurities involved in the usual methods, especially in the transition $h\to 0$, are discussed. Consequently, the author gives, following Ludwig [Z. Physik **135** (1953), 483–511; MR **15**, 378], some restrictions imposed on the quantum states of systems which allow for a classical description. The fulfilment of these conditions enables one to define the usual macroscopic observables satisfying the classical equations of motion. The methods proposed by the author are compared with those of other authors. *J. Werle* (Warsaw).

Syozi, Itiro, and Nakano, Huzio. Statistical models of ferrimagnetism. Progr. Theoret. Phys. **13** (1955), 69–78.

The method given by Yang for deriving the exact spontaneous magnetization of a two-dimensional Ising model of ferromagnetism is here applied to determine the same quantity as a function of temperature for a few two-dimensional Ising models of ferrimagnets. The first model considered is a square lattice with two kinds of spins differing by the values of their magnetic moment, neighboring sites being always occupied by different spins, and with antiferromagnetic exchange coupling between nearest neighbors. The other models correspond to "decorated" lattices. The decorated lattice corresponding to a given lattice is obtained by inserting a site between any pair of nearest neighbors of the original lattice. One puts spins of one type in the sites of the original lattice and spins of another type in the additional sites. Exchange coupling is taken between pairs of different spins in neighbouring positions. An additional coupling can also be considered between neighboring spins of the origin lattice. The partition function of a decorated lattice (with two types of spin) is shown to be simply related to that of the original lattice (with one type of spin). The magnetization curves obtained have the same general shape as derived from calculations based on the Weiss internal field approximation. *L. Van Hove* (Utrecht).

See also: Pignedoli, p. 168.

s of
The
the
sical
the

).

s of
55),

xact
sing
the
wo-
odel
pins
ent,
rent
be-
d to
ding
ween
One
ttice
ange
s in
also
rigin
ttice
ed to
The
neral
Weiss
t).